Existence of positive solutions of discrete delayed equations

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We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$ where $s = -\infty$ and $q = \infty$ are admitted, too. The subject of our study is a linear scalar discrete equation of k-th order

$$\Delta x(n) = -\sum_{i=0}^{k} p_i(n) x(n-i), \tag{1}$$

where $p_0 \colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $p_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}_+ := [0, \infty)$, $i = 1, \dots, k$, $k \ge 1$, a is an integer, and $n \in \mathbb{Z}_a^\infty$.

Theorem 1. *Let $\sum_{i=1}^k p_i(n) > 0$ for any $n \in \mathbb{Z}_a^\infty$. Then, for the existence of a solution $x \colon \mathbb{Z}_{a-k}^\infty \to \mathbb{R}^+ := (0, \infty)$ of (1), the existence of a function $\nu \colon \mathbb{Z}_{a-k}^\infty \to \mathbb{R}^+$ such that*

$$\Delta \nu(n) \le -\sum_{i=0}^{k} p_i(n) \nu(n-i)$$

for $n \in \mathbb{Z}_a^{+\infty}$ is sufficient and necessary.

Definition 1. Let us define the expression $\ln_q n$, $q \ge 1$ by the formula $\ln_q n = \ln(\ln_{q-1} n)$, $\ln_0 n \equiv n$ where $n > \exp_{q-2} 1$ and $\exp_s n = \exp\big(\exp_{s-1} n\big)$, $s \ge 1$, $\exp_0 n \equiv n$ and $\exp_{-1} n \equiv 0$ (with $\ln_0 n$, $\ln_1 n$ abbreviated to n, $\ln n$ in the sequel).

Let $\ell \ge 0$ be a fixed integer. We define an auxiliary function

$$p_\ell(n) = \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \dots + \frac{k}{8(n \ln n \dots \ln_\ell n)^2}\right]$$

which plays an important role in the investigation of positive solutions of an equation

$$\Delta x(n) = -p(n) x(n-k) \tag{2}$$

being a particular case of (1). We assume that n is sufficiently large such that p_ℓ is well defined. Applying Theorem 1 to (2), we get:

Theorem 2. *Let $\ell \ge 0$ be a fixed integer and $0 < p(n) \le p_\ell(n)$ for $n \to +\infty$. Then the equation (2) has a positive solution $x = x(n)$ provided that n is sufficiently large.*

References

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