On exponential equivalence of solutions to nonlinear ordinary differential equations.

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1 Introduction

In the first part of the paper the equations

$$y^{(n)} + \frac{a}{x^2}y + p(x)y|y|^{k-1} = f(x), \tag{1}$$

$$z^{(n)} + \frac{a}{x^2}z + p(x)z|z|^{k-1} = 0 \tag{2}$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$ are considered. Functions $p(x)$ and $f(x)$ are assumed to be continuous as $x > x_0 > 0$, $p(x) \not\equiv 0$. Exponential equivalence of solutions to equations (1), (2) is proved under some assumptions on the function $f(x)$.

If $a = 0$ equation (2) is well-known Emden–Fowler equation:

$$z^{(n)} + p(x)z|z|^{k-1} = 0.$$

A lot of results on the asymptotic behaviour of solutions to this equation and its generalizations were obtained in [1, 2, 3, 4, 6]. Note that equation (2) with $a \neq 0$ can't be reduced to Emden–Fowler differential equation by any substitution of dependent or independent variables.

In the second part of the paper the equations

$$y^{[n]} + p_0|y|^{k-1}y = f(x), \tag{3}$$

$$y^{[n]} + p_0|y|^{k-1}y = g(x), \tag{4}$$

where

$$y^{[n]} = r_n(x)\frac{d}{dx}\left(\cdots\frac{d}{dx}\left(r_1(x)\frac{d}{dx}\left(r_0(x)y\right)\right)\cdots\right), \tag{5}$$

$k > 1$, $p_0 = \text{const} \neq 0$ and the functions $r_m(x)$, $m = 0, \ldots, n$, $f(x)$, $g(x)$ are continuous on $(x_0, +\infty)$, $x_0 > 0$. Exponential equivalence of solutions to equations (3), (4) is proved under some assumptions.

2 Exponential equivalence of solutions to nonlinear differential equations

Consider differential equations

$$y^{(n)} + \frac{a}{x^2}y + p(x)y|y|^{k-1} = e^{-\alpha x}f(x), \tag{6}$$

$$z^{(n)} + \frac{a}{x^2}z + p(x)z|z|^{k-1} = e^{-\alpha x}g(x). \tag{7}$$

with $n \geq 2$, $k > 1$, $a \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$

Lemma 1 ([5]) *If function $y(x)$ and its n-th derivative $y^{(n)}(x)$ tend to zero as $x \to +\infty$ than the same holds for $y^{(j)}(x)$, $0 < j < n$.*

Lemma 2 *Let $y(x)$ be a solution to equation (6) such that $y(x)$ tends to zero as $x \to +\infty$. Then it holds*

$$y(x) = \mathbf{J^n}\left[e^{-\alpha x}f(x) - \frac{a}{x^2}y(x) - p(x)\left[y(x)\right]^k_{\pm}\right]$$

with $\left[y(x)\right]^k_{\pm} = |y|^{k-1}y$. $\mathbf{J}$ is the operator that maps tending to zero as $x \to +\infty$ function $\varphi(x)$ to its antiderivative:

$$\mathbf{J}[\varphi](x) = -\int_x^{+\infty} \varphi(t)\,dt.$$

Theorem 1 *Let $p(x)$, $f(x)$, $g(x)$ be continuous bounded functions defined as $x > x_0 > 0$, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation* (6) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (7) *such that*

$$|z(x) - y(x)| = O\left(e^{-\alpha x}\right), \quad x \to +\infty.$$

Remark 1 *Obviously, equations* (6) *and* (7) *in Theorem 1 can be swapped.*

Back to equations (1), (2):

$$y^{(n)} + \frac{a}{x^2} y + p(x) y |y|^{k-1} = f(x),$$

$$z^{(n)} + \frac{a}{x^2} z + p(x) z |z|^{k-1} = 0$$

with $k > 1$, $a \in \mathbb{R} \setminus \{0\}$.

Corollary 1.1 *Suppose continuous function $f(x)$ satisfies the following condition:*

$$f(x) = O(e^{-\alpha x}), \quad \alpha > 0.$$

Let function $p(x)$ be a continuous bounded function, $p(x) \not\equiv 0$. Then for any solution $y(x)$ to equation (1) *that tends to zero as $x \to +\infty$ there exists a unique solution $z(x)$ to equation* (2) *such that*

$$|y(x) - z(x)| = O\left(e^{-\alpha x}\right), \quad x \to +\infty.$$

3 Exponential equivalence of solutions to nonlinear differential equations with quasiderivative

Consider differential equation (3), (4)

$$y^{[n]} + p_0 |y|^{k-1} y = f(x),$$

$$y^{[n]} + p_0 |y|^{k-1} y = g(x),$$

with $n \geq 2$, $k > 1$ and $p_0 = \text{const} \neq 0$,

$$y^{[n]} = r_n(x) \frac{d}{dx}\left(\cdots \frac{d}{dx}\left(r_1(x) \frac{d}{dx}\left(r_0(x) y \right) \right) \cdots \right).$$

Remark 2 *In [4] some sufficient conditions are given for the differential operator*

$$L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} q_j(x) \frac{d^j}{dx^j}$$

to be represented as quasiderivative (5).

Theorem 2 *Suppose for each $m = 0,\ 1, \ldots,\ n$ the function $r_m(x)$ is constant or tends to infinity as $x \to +\infty$, and there exists $b = \text{const} > 0$ such that*

$$\int_{x_0}^{+\infty} \left(f(x) - g(x)\right)^2 \frac{e^{2bx}}{r_n^2(x)} < \infty, \ x_0 > 0.$$

Then for any solution $y(x)$ to equation (3) *such that this solution and its first derivative tend to zero as $x \to +\infty$ there exists a solution $\tilde{y}(x)$ to equation* (4) *satisfying the following conditions:*

$$|y(x) - \tilde{y}(x)| = o(e^{-bx}), \ x \to +\infty, \quad \int_{x_0}^{+\infty} \left(y(x) - \tilde{y}(x)\right)^2 e^{2bx} < \infty.$$

References

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