On some estimates for the first eigenvalue of a Sturm-Liouville problem with Dirichlet boundary conditions and a weighted integral condition

S. Ezhak, M. Telnova

Plekhanov Russian University of Economics,
svetlana.ezhak@gmail.com, mytelnova@yandex.ru

1 Introduction

We consider a problem which origin was the Lagrange problem of finding the form of the firmest column of given volume. The Lagrange problem was the source for different extremal eigenvalue problems. Some of them are eigenvalue problems for second-order differential equations with integral conditions on the potential.

The Dirichlet problem for the equation $y'' + \lambda Q(x)y = 0$ with a non-negative summable on $[0,1]$ function Q satisfying the condition $\int_0^1 Q^\gamma(x)dx = 1$, as $\gamma \in \mathbb{R}, \gamma \neq 0$, was considered in [1]. The Dirichlet problem for the equation $y'' - Q(x)y + \lambda y = 0$ with a real integrable on $(0,1)$ by Lebesgue function Q was considered in [2] for $\gamma \geqslant 1$. In this paper we consider a problem of that kind in the case when the integral condition contains a weight function.

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real–valued locally integrable functions on $(0,1)$ with non–negative values such that the following integral condition holds

$$\int_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)dx = 1, \quad \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0. \tag{3}$$

A function y is a *solution* to problem (1), (2) if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

We give estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q), \quad M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

For any function $Q \in T_{\alpha,\beta,\gamma}$ by H_Q we denote the closure of the set $C_0^\infty(0,1)$ in the norm

$$\|y\|_{H_Q} = \left(\int_0^1 {y'}^2 dx + \int_0^1 Q(x) y^2 dx \right)^{\frac{1}{2}}.$$

For any function $Q \in T_{\alpha,\beta,\gamma}$ we can prove (see, for example, [7], [9]) that

$$\lambda_1(Q) = \inf_{y \in H_Q \setminus \{0\}} R[Q,y], \quad \text{where } R[Q,y] = \frac{\int_0^1 \left({y'}^2 - Q(x) y^2 \right) dx}{\int_0^1 y^2 dx}.$$

2 Some estimates for the first eigenvalue of a Sturm-Liouville problem with a weighted integral condition

For problem (1)–(3) for $\alpha = \beta = 0$ the following theorem was proved (see, for example, [4] – [6]).

Theorem 1 *1. If $\gamma > 1$ then $m_{0,0,\gamma} \geqslant \frac{\pi^2}{2}$, $M_{0,0,\gamma} = \pi^2$ and there exist functions $Q_* \in T_{0,0,\gamma}$ and $u \in H_0^1(0,1)$ such that $m_{0,0,\gamma} = R[Q_*, u] \geqslant \frac{\pi^2}{2}$.*

2. *If $\gamma = 1$ then $M_{0,0,1} = \pi^2$, $m_{0,0,1} = \lambda_*$, where $\lambda_* \in (0, \pi^2)$ is the solution to the equation $2\sqrt{\lambda} = tg\left(\frac{\sqrt{\lambda}}{2}\right)$. Here $m_{0,0,1}$ is attained at $Q(x) = \delta\left(x - \frac{1}{2}\right)$.*

3. *If $1/2 \leqslant \gamma < 1$ then $m_{0,0,\gamma} = -\infty$, $M_{0,0,\gamma} = \pi^2$.*

4. *If $1/3 \leqslant \gamma < 1/2$ then $m_{0,0,\gamma} = -\infty$, $M_{0,0,\gamma} \leqslant \pi^2$.*

5. *If $0 < \gamma < 1/3$ then $m_{0,0,\gamma} = -\infty$, $M_{0,0,\gamma} < \pi^2$.*

6. *If $\gamma < 0$ then $m_{0,0,\gamma} = -\infty$, $M_{0,0,\gamma} < \pi^2$, and there exist functions $Q_* \in T_{0,0,\gamma}$ and $u \in H_0^1(0,1)$ such that $M_{0,0,\gamma} = R[Q_*, u]$.*

Remark 1 *The result $M_{0,0,\gamma} < \pi^2$ for $0 < \gamma < 1/2$ was obtained in [3].*

The following theorem gives some results for arbitrary real numbers α, β.

Theorem 2 1. *For any α, β, γ, $\gamma \neq 0$, we have $M_{\alpha,\beta,\gamma} \leqslant \pi^2$.*

2. *If $\gamma < 0$ or $0 < \gamma < 1$ then $m_{\alpha,\beta,\gamma} = -\infty$.*

3.1 *If $\gamma = 1$ and $\alpha, \beta \leqslant 0$ then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*

3.2 *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$ then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

3.3 *if $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$ then $-\pi^2 \leqslant m_{\alpha,\beta,\gamma} \leqslant \pi^2$;*

3.4 *if $\gamma > 1$ and $0 < \alpha$, $\beta \leqslant 2\gamma - 1$ then $m_{\alpha,\beta,\gamma} \geqslant \left(1 - 2^{\frac{3\gamma-2}{\gamma}}\left(\frac{2\gamma-1}{\gamma}\right)^{\frac{2\gamma-1}{\gamma}}\right)\pi^2$;*

3.5 *if $\gamma > 1$ and $\beta \leqslant 0 < \alpha \leqslant 2\gamma - 1$ or $\alpha \leqslant 0 < \beta \leqslant 2\gamma - 1$ then $m_{\alpha,\beta,\gamma} \geqslant \left(1 - \left(\frac{2\gamma-1}{\gamma}\right)^{\frac{2\gamma-1}{\gamma}}\right)\pi^2$;*

3.6 *if $\gamma > 1$ and $\alpha, \beta \leqslant 0$ then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

Proof

1. Let α, β, γ be arbitrary real numbers, $\gamma \neq 0$. Let us prove that $M_{\alpha,\beta,\gamma} \leqslant \pi^2$.

Consider the function

$$v_\varepsilon(x) = \begin{cases} 0, & 0 < x < \varepsilon\,; \\ 1, & \varepsilon \leqslant x \leqslant 1 - \varepsilon\,; \\ 0, & 1 - \varepsilon < x < 1\,, \end{cases}$$

where $0 < \varepsilon < 1$.

By average processing we obtain the function $y_\varepsilon(x) = v_{\varepsilon\rho}(x) \cdot \sin \pi x$ of $C_0^\infty(0,1)$, $\rho < \varepsilon$.

Since $\|y_\varepsilon(x) - \sin \pi x\|_{H_0^1(0,1)} \to 0$ as $\varepsilon \to 0$ and

$$\inf_{y \in H_0^1(0,1)} V[y] = \inf_{y \in H_0^1(0,1)} \frac{\int_0^1 {y'}^2 dx}{\int_0^1 y^2} = \pi^2$$

then for any ε we have

$$V[y_\varepsilon] = \frac{\int_0^1 {y_\varepsilon'}^2 dx}{\int_0^1 y_\varepsilon^2\, dx} \geqslant \pi^2 \quad \text{ and } \quad V[y_\varepsilon] \to \pi^2 \quad \text{as} \quad \varepsilon \to 0.$$

Then for any function $Q \in T_{\alpha,\beta,\gamma}$ we have $R[Q, y_\varepsilon] < V[y_\varepsilon] = \frac{\int_0^1 {y_\varepsilon'}^2 dx}{\int_0^1 y_\varepsilon^2\, dx}$. Consequently, for any function $Q \in T_{\alpha,\beta,\gamma}$ we obtain

$$\inf_{y \in H_Q \setminus \{0\}} R[Q, y] \leqslant \lim_{\varepsilon \to 0} R[Q, y_\varepsilon] \leqslant \lim_{\varepsilon \to 0} V[y_\varepsilon] = \pi^2,$$

and therefore,

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_Q \setminus \{0\}} R[Q, y] \leqslant \pi^2.$$

2. Let us prove that if $\gamma<0$ or $0<\gamma<1$ then $m_{\alpha,\beta,\gamma}=-\infty$.

2.1) Suppose that $\gamma<0$, $\alpha,\beta>0$. Put $\alpha\geqslant\beta$. Consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x)=\begin{cases}\left(\dfrac{1-\varepsilon^{2\alpha}(1-\varepsilon)^{\alpha}}{2\varepsilon}\right)^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & 0<x<\varepsilon\,;\\ \left(\dfrac{\varepsilon^{2\alpha}(1-\varepsilon)^{\alpha}}{1-2\varepsilon}\right)^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & \varepsilon\leqslant x\leqslant 1-\varepsilon;\\ \left(\dfrac{1-\varepsilon^{2\alpha}(1-\varepsilon)^{\alpha}}{2\varepsilon}\right)^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & 1-\varepsilon<x<1,\end{cases}$$

where $0<\varepsilon<1$. Then

$$\int_{\varepsilon}^{1-\varepsilon}y^2dx\leqslant\int_{\varepsilon}^{1-\varepsilon}\frac{x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\alpha}{\gamma}}}{\varepsilon^{-\frac{\alpha}{\gamma}}(1-\varepsilon)^{-\frac{\alpha}{\gamma}}}(1-x)^{\frac{\alpha-\beta}{\gamma}}y^2dx=$$
$$=\frac{\varepsilon^{-\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{-\frac{1}{\gamma}}}\int_{\varepsilon}^{1-\varepsilon}\frac{\varepsilon^{\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{\frac{1}{\gamma}}}\cdot\frac{x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}}{\varepsilon^{-\frac{\alpha}{\gamma}}(1-\varepsilon)^{-\frac{\alpha}{\gamma}}}y^2dx=$$
$$=\frac{\varepsilon^{-\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{-\frac{1}{\gamma}}}\int_{\varepsilon}^{1-\varepsilon}Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2dx.$$

For any $y\in H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ by the the Hölder inequality we have

$$\int_0^1y^2dx=\int_0^{\varepsilon}y^2dx+\int_{\varepsilon}^{1-\varepsilon}y^2dx+\int_{1-\varepsilon}^1y^2dx\leqslant$$
$$\leqslant\frac{\varepsilon^2}{2}\int_0^{\varepsilon}y'^2dx+\frac{\varepsilon^{-\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{-\frac{1}{\gamma}}}\int_{\varepsilon}^{1-\varepsilon}Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2dx+\frac{\varepsilon^2}{2}\int_{1-\varepsilon}^1y'^2dx.$$

Then

$$\int_0^1Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2dx\geqslant\int_{\varepsilon}^{1-\varepsilon}Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2dx\geqslant\left(\int_0^1y^2dx-\frac{\varepsilon^2}{2}\int_0^1y'^2dx+\frac{\varepsilon^2}{2}\int_{\varepsilon}^{1-\varepsilon}y'^2dx\right)\frac{\varepsilon^{\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{\frac{1}{\gamma}}}.$$

For any ε the function $y_*=\sin\pi x$ belongs to the space $H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ and

$$\int_0^1{y_*}^2dx=\frac{1}{2},\quad\int_0^1{y_*}'^2dx=\frac{\pi^2}{2},\quad\int_{\varepsilon}^{1-\varepsilon}{y_*}'^2dx=\frac{\pi^2}{2}(1-2\varepsilon)-\frac{\pi}{2}\sin 2\pi\varepsilon,$$

Thus

$$\inf_{y\in H_{Q_{\varepsilon,\alpha,\beta,\gamma}}\setminus\{0\}}R[Q_{\varepsilon,\alpha,\beta,\gamma},y]\leqslant R[Q_{\varepsilon,\alpha,\beta,\gamma},y_*]=\pi^2-\left(1-\varepsilon^3\pi^2-\frac{\pi}{2}\varepsilon^2\sin 2\pi\varepsilon\right)\cdot\frac{\varepsilon^{\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{\frac{1}{\gamma}}}.$$

Therefore,

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}}R[Q,y]\leqslant\lim_{\varepsilon\to0}\inf_{y\in H_{Q_{\varepsilon,\alpha,\beta,\gamma}}\setminus\{0\}}R[Q_{\varepsilon,\alpha,\beta,\gamma},y]\leqslant\lim_{\varepsilon\to0}R[Q_{\varepsilon,\alpha,\beta,\gamma},y_*]=-\infty.$$

If $\beta\geqslant\alpha$ then on the segment $[\varepsilon,1-\varepsilon]$ we define the function $Q_{\varepsilon,\alpha,\beta,\gamma}$ as follows

$$Q_{\varepsilon,\alpha,\beta,\gamma}=\left(\frac{\varepsilon^{2\beta}(1-\varepsilon)^{\beta}}{1-2\varepsilon}\right)^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}=\frac{\varepsilon^{\frac{\beta}{\gamma}}}{(1-2\varepsilon)^{\frac{1}{\gamma}}}\cdot\frac{x^{-\frac{\beta}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}}{\varepsilon^{-\frac{\beta}{\gamma}}(1-\varepsilon)^{-\frac{\beta}{\gamma}}}\cdot x^{\frac{\beta-\alpha}{\gamma}},$$

similarly we obtain $m_{\alpha,\beta,\gamma}=-\infty$.

2.2) Suppose that $\gamma<0$, $\beta\leqslant0<\alpha$. Consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x)=\begin{cases}(\alpha+1)^{\frac{1}{\gamma}}\,\varepsilon^{-\frac{\alpha+1}{\gamma}}(1-\varepsilon)^{\frac{1}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & 0<x\leqslant\varepsilon\,;\\ (\alpha+1)^{\frac{1}{\gamma}}\,\varepsilon^{\frac{1}{\gamma}}(1-\varepsilon^{\alpha+1})^{-\frac{1}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & \varepsilon<x<1,\end{cases}$$

where $0 < \varepsilon < 1$.

For any function $y \in H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ by the the Hölder inequality we obtain

$$\int_0^1 y^2dx = \int_0^\varepsilon y^2dx + \int_\varepsilon^1 y^2dx \leqslant \frac{\varepsilon^2}{2}\int_0^\varepsilon y'^2dx + \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\gamma}} \varepsilon^{-\frac{1}{\gamma}}(1-\varepsilon^{\alpha+1})^{\frac{1}{\gamma}}\int_\varepsilon^1 Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2dx.$$

Then

$$\int_\varepsilon^1 Q_{\varepsilon,\alpha,\gamma}(x)y^2dx \geqslant (\alpha+1)^{\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}}(1-\varepsilon^{\alpha+1})^{-\frac{1}{\gamma}}\left(\int_0^1 y^2dx - \frac{\varepsilon^2}{2}\int_0^1 y'^2dx\right).$$

For any ε the function $y_* = (1-x)^{\frac{\beta}{\gamma}}\sin\pi x$ belongs to the space $H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ and

$$R[Q_{\varepsilon,\alpha,\gamma}, y_*] \leqslant \frac{\int_0^1 {y_*}'^2dx + (\alpha+1)^{\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}}(1-\varepsilon^{\alpha+1})^{-\frac{1}{\gamma}}(\frac{\varepsilon^2}{2}\cdot\int_0^1 {y_*}'^2dx - \int_0^1 {y_*}^2dx)}{\int_0^1 {y_*}^2dx}.$$

Therefore,

$$m_{\alpha,\beta,\gamma} = \inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \lim_{\varepsilon\to 0}\inf_{y\in H_{Q_{\varepsilon,\alpha,\beta,\gamma}}\setminus\{0\}} R[Q_{\varepsilon,\alpha,\beta,\gamma}, y] \leqslant \lim_{\varepsilon\to 0} R[Q_{\varepsilon,\alpha,\beta,\gamma}, y_*] = -\infty.$$

Note that the case $\gamma < 0$, $\alpha \leqslant 0 < \beta$ is symmetrical with the case $\gamma < 0$, $\beta \leqslant 0 < \alpha$.

2.3) Suppose that $\gamma < 0$, $\alpha, \beta \leqslant 0$. Consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \begin{cases} (1-\varepsilon)^{\frac{1}{\gamma}}\varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & 0 < x \leqslant \varepsilon\,; \\ (1-\varepsilon)^{-\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & \varepsilon < x < 1, \end{cases}$$

where $0 < \varepsilon < 1$. For any ε the function $y_* = x^{\frac{\alpha}{\gamma}}(1-x)^{\frac{\beta}{\gamma}}\sin\pi x$ belongs to the space $H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ and similarly to the cases 2.1) and 2.2) we obtain $m_{\alpha,\beta,\gamma} = -\infty$.

2.4) Suppose that $0 < \gamma < 1$ and α, β are arbitrary real numbers. For $0 < \varepsilon < 1$ consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \begin{cases} 0, & 0 \leqslant x < \frac{1}{2} - \frac{\varepsilon}{2}; \\ \varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}\,(1-x)^{-\frac{\beta}{\gamma}}, & \frac{1}{2} - \frac{\varepsilon}{2} \leqslant x \leqslant \frac{1}{2} + \frac{\varepsilon}{2}; \\ 0, & \frac{1}{2} + \frac{\varepsilon}{2} < x \leqslant 1. \end{cases}$$

For any ε the function $y_* = \sin\pi x$ belongs to the space $H_{Q_{\varepsilon,\alpha,\beta,\gamma}}$ and for any $\varepsilon, \alpha, \beta, \gamma$ there exists a constant $C = \min_{[\frac{1}{2}-\frac{\varepsilon}{2}, \frac{1}{2}+\frac{\varepsilon}{2}]} x^{-\frac{\alpha}{\gamma}}\,(1-x)^{-\frac{\beta}{\gamma}}$ such that

$$\int_0^1 Q_{\varepsilon,\alpha,\beta,\gamma}(x){y_*}^2dx \geqslant \int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}\,(1-x)^{-\frac{\beta}{\gamma}}\sin^2\pi x dx \geqslant$$

$$\geqslant C\cdot\varepsilon^{-\frac{1}{\gamma}}\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \frac{1-\cos 2\pi x}{2}dx = C\cdot\varepsilon^{-\frac{1}{\gamma}}\left(\frac{\varepsilon}{2} + \frac{\sin\pi\varepsilon}{2\pi}\right).$$

similarly to the cases 2.1)— 2.3) we obtain $m_{\alpha,\beta,\gamma} = -\infty$.

3.1. Suppose that $\gamma = 1$ and $\alpha, \beta \leqslant 0$. It was proved (see, for example, [1]) that for any function $y \in H_0^1(0,1)$ the inequality $\sup\limits_{[0,1]} y^2 \leqslant \frac{1}{4}\int_0^1 y'^2dx$ holds. For any function $Q \in T_{\alpha,\beta,\gamma}$ and for any function $y \in H_Q$ we obtain

$$\int_0^1 Q(x)y^2dx \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Q(x)x^\alpha dx \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Q(x)x^\alpha(1-x)^\beta dx \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha} \leqslant \sup_{[0,1]} y^2 \leqslant \frac{1}{4}\int_0^1 y'^2dx.$$

Therefore,

$$m_{\alpha,\beta,\gamma} = \inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \geqslant \frac{3}{4}\inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_Q\setminus\{0\}}\frac{\int_0^1 y'^2dx}{\int_0^1 y^2dx} \geqslant \frac{3}{4}\inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 y'^2dx}{\int_0^1 y^2dx} = \frac{3}{4}\pi^2.$$

3.2. Suppose that $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$. For any function $Q \in T_{\alpha,\beta,\gamma}$ we have

$$\int_0^1 Q(x)y^2dx = \int_0^1 Q(x)y^2x^{\alpha}x^{-\alpha}dx \leqslant \sup_{[0,1]} \frac{y^2}{x^{\alpha}} \int_0^1 Q(x)x^{\alpha}(1-x)^{\beta}dx \leqslant \sup_{[0,1]} \frac{y^2}{x}.$$

For any $x \in (0,1)$ by the the Hölder inequality we obtain

$$y^2(x) = \left(\int_0^x y'(t)dt\right)^2 \leqslant x \int_0^x y'^2(t)dt \leqslant x \int_0^1 y'^2(t)dt \quad \text{and} \quad \sup_{[0,1]} \frac{y^2}{x} \leqslant \int_0^1 y'^2dx.$$

Then

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_Q \setminus \{0\}} \frac{\int_0^1 y'^2 - \int_0^1 Q(x)y^2dx}{\int_0^1 y^2dx} \geqslant 0.$$

Note that the case $\alpha \leqslant 0 < \beta \leqslant 1 = \gamma$ is symmetrical with the case $\beta \leqslant 0 < \alpha \leqslant 1 = \gamma$. Similarly we can prove the results of 3.3) — 3.6).

References

[1] Yu. V. Egorov, V. A. Kondratiev, *On Spectral theory of elliptic operators.* – in Operator theory: Advances and Applications. Birkhouser (1996), 1–325.

[2] V. A. Vinokurov, V. A. Sadovnichii, *On the Range of Variation of an Eigenvalue When the Potential Is Varied.* Doklady Mathematics Translated from Doklady Akademii Nauk, Vol. 68, No. 2, (2003), 247–253. (Russian)

[3] A. A. Vladimirov, *On some a priori majorant of eigenvalues of Sturm–Liouville problems.* arXiv preprint arXiv:1602.05228, 2016 - arxiv.org

[4] S. S. Ezhak, *On estimates for the first eigenvalue of the Sturm - Liouville problem with Dirichlet boundary conditions.* In: Astashova I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, M.: UNITY-DANA, (2012), 517–559. (Russian)

[5] S. S. Ezhak, *On estimates for the first eigenvalue of the Sturm - Liouville problem with Dirichlet boundary conditions and integral condition.* Differential and Difference Equations with Applications, Springer Proceedings in Mathematics and Statistics, Springer New York LLC, V. 47, (2013), 387–394.

[6] S. S. Ezhak, *On a minimization problem for a functional generated by the Sturm - Liouville problem with integral condition on the potential.* Vestnik of Samara State University, No 6 (128), (2015), 57–61. (Russian)

[7] M. Yu. Telnova, *Some estimates for the first eigenvalue of the Sturm–Liouville problem with a weight integral condition.* Mathematica Bohemica, V. 137 (2),(2012), 229–238.

[8] I. V. Astashova, *Qualitative properties of solutions to quasilinear ordinary differential equations.* In: Astashova I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, M.: UNITY-DANA, (2012), 22–290. (Russian)

[9] M. Yu. Telnova, *Estimates for the first eigenvalue of a Sturm–Liouville problem with Dirichlet conditions and a weight integral condition.* In: Astashova I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, M.: UNITY-DANA, (2012), 608–647. (Russian)