Monotone and Oscillatory Solutions of Higher Order Nonlinear Functional Differential Systems

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On an infinite interval $[a,+\infty[$, the functional differential system

$$u_1^{(n_1)}(t)=f_1\big(t,u_2(\tau_1(t))\big),\quad u_2^{(n_2)}(t)=f_2\big(t,u_1(\tau_2(t))\big) \tag{1}$$

is considered, where $n_1 \geq 1$, $n_2 \geq 2$, $a > 0$, while $f_i : [a,+\infty[\times\mathbb{R}\to\mathbb{R}$ and $\tau_i : [a,+\infty[\to\mathbb{R}$ $(i=1,2)$ are continuous functions. Moreover,

$$\lim_{t\to+\infty}\tau_i(t)=+\infty \ \ (i=1,2),$$

and one of the following two conditions

$$f_i(t,0)=0,\quad (-1)^{i-1}f_i(t,x)\leq(-1)^{i-1}f_i(t,y) \ \text{ for } \ t>a,\ \ x<y \ \ (i=1,2); \tag{2}$$

$$f_i(t,0)=0,\quad f_i(t,x)\leq f_i(t,y) \ \text{ for } \ t\geq a,\ \ x<y \ \ (i=1,2) \tag{3}$$

is satisfied.

Asymptotic (including oscillatory) properties of solutions of system (1) previously have been investigated mainly in the cases where this system can be reduced to one n_1+n_2-order functional differential equation, or in the cases where $n_1=n_2=1$ (see, [1, 2] and the references therein). The results below concern the case, where $n_1+n_2>2$, $\tau_i(t)\not\equiv t$ $(i=1,2)$, and system (1) cannot be reduced to one equation.

Let $a_0 \geq a$. A vector function $(u_1,u_2) : [a_0,+\infty[\to\mathbb{R}$ is said to be a **solution of system** (1) if u_1 and u_2 are, respectively, n_1-times and n_2-times continuously differentiable functions, and there exist continuous functions $v_i :]-\infty,a_0]\to\mathbb{R}$ $(i=1,2)$ such that on $[a_0,+\infty[$ equalities (1) are fulfilled, where

$$u_i(t)=v_i(t) \ \text{ for } \ t\leq a_0 \ \ (i=1,2).$$

A solution (u_1,u_2) of system (1), defined on some interval $[a_0,+\infty[\subset[a,+\infty[$, is said to be **proper** if it is not identically zero in any neighborhood of $+\infty$.

A proper solution of system (1) is said to be **oscillatory** if at least one of its components changes the sign in any neighborhood of $+\infty$.

Note that if one of conditions (2) and (3) is satisfied, then both components of every oscillatory solution of system (1) change the sign in any neighborhood of $+\infty$.

A nontrivial solution $(u_1,u_2) : [a_0,+\infty[\to\mathbb{R}$ of system (1) is said to be a **Kneser solution** if on $[a_0,+\infty[$ it satisfies the inequalities

$$(-1)^i u_1^{(i)}(t)u_1(t)\geq 0 \ \ (i=1,\dots,n_1),$$
$$(-1)^k u_2^{(k)}(t)u_2(t)\geq 0 \ \ (k=1,\dots,n_2),$$

and it is said to be **rapidly increasing** if

$$\lim_{t\to+\infty} |u_i^{(n_i-1)}(t)| = +\infty \quad (i=1,2).$$

Let

$$n = n_1 + n_2,$$

and following I. Kiguradze [1, 2] introduce the definitions.

Definition 1 *System* (1) *has the* ***property*** A_0 *if every its proper solution for* n *even is oscillatory, and for* n *odd either is oscillatory or is a Kneser solution.*

Definition 2 *System* (1) *has the* ***property*** B_0 *if every its proper solution for* n *even either is oscillatory, or is a Kneser solution, or is rapidly increasing, and for* n *odd either is oscillatory or is rapidly increasing.*

I. Kiguradze [1, 2] has established unimprovable in a certain sense conditions under which the differential system

$$u_1^{(n_1)}(t) = f_1(t, u_2(t)), \quad u_2^{(n_2)}(t) = f_2(t, u_1(t))$$

has the property A_0 (the property B_0). The theorems below are the generalizations of those results for system (1).

If m is a natural number, then by $\mathcal{N}_m^0$ we denote the set of those $k \in \{1, \dots, m\}$ for which $m + k$ is even.

For any natural k, we put

$$\varphi_k(t,x) = x\left[|\tau_2(t)|^{n_1-1} + \int_a^{\tau_2(t)} (\tau_2(t)-s)^{n_1-1}\big|f_1\big(t, x|\tau_1(s)|^{k-1}\big)\big|\,ds\right].$$

Theorem 1 *Let condition* (2) *hold and let for any* $x \neq 0$ *and* $k \in \mathcal{N}_{n_2-1}^0$ *the equalities*

$$\int_a^{+\infty} |f_1(t,x)|\,dt = +\infty, \quad \int_a^{+\infty} t^{n_2-1}|f_2(t,x)|\,dt = +\infty, \tag{4}$$

$$\int_a^{+\infty} t^{n_2-k-1}\big|f_2(t,\varphi_k(t,x))\big|\,dt = +\infty \tag{5}$$

be satisfied. Then system (1) *has the property* A_0.

Theorem 2 *Let* $n_2 > 2$ ($n_2 = 2$) *and condition* (3) *hold. If, moreover, for any* $x \neq 0$ *and* $k \in \mathcal{N}_{n_2-2}^0$ *equalities* (4) *and* (5) *are satisfied* (*for any* $x \neq 0$ *equalities* (4) *are satisfied*), *then system* (1) *has the property* B_0.

Remark 1 *For equality* (5) *to be satisfied for any* $x \neq 0$ *and* $k \in \mathcal{N}^0_{n_2-1}$ *it is sufficient that the equality*

$$\int\limits_a^{+\infty} \big|f_2\big(t, x|\tau_2(t)|^{n_1-1}\big)\big|\,dt = +\infty$$

be satisfied for any $x \neq 0$*.*

The conditions of Theorems 1 and 2 do not guarantee the existence of proper solutions appearing in the definitions of the properties A_0 and B_0. The problem on the existence of such solutions needs additional investigation.

The following theorems are valid.

Theorem 3 *Let* $n_1 + n_2$ *be even and along with* (2) *the condition*

$$\tau_i(t) < t, \quad f_i(t,x) \neq 0 \ \text{ for } \ t \geq a, \quad x \neq 0 \ \ (i = 1, 2) \tag{6}$$

be satisfied. If, moreover, for any $x \neq 0$ *and* $k \in \mathcal{N}^0_{n_2-1}$ *equalities* (4) *and* (5) *are fulfilled, then system* (1) *has an infinite set of proper solutions and every such solution is oscillatory.*

Theorem 4 *Let* $n_1 + n_2$ *be odd and conditions* (3) *and* (6) *hold. If, moreover,* $n_2 > 2$ ($n_2 = 2$) *and for any* $x \neq 0$ *and* $k \in \mathcal{N}^0_{n_2-2}$ *equalities* (4) *and* (5) *are satisfied* (*for any* $x \neq 0$ *equalities* (4) *are satisfied*)*, then system* (1) *has infinite sets of oscillatory and rapidly increasing solutions.*

References

[1] I. Kiguradze, On oscillatory solutions of higher order nonlinear nonautonomous differential equations and systems. *Czech-Georgian Workshop on Boundary Value Problems – WBVP-2016, Brno, Czech Republic*, `http://users.math.cas.cz/ sremr/wbvp2016/abstracts/kiguradze1.pdf`.

[2] I. Kiguradze, Oscillatory solutions of higher order nonlinear nonautonomous differential systems. *Mem. Differential Equations Math. Phys.* **69** (2016), 123–127.