On Boundary Value Problems on an Infinite Interval for Higher Order Nonlinear Differential Systems

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On an infinite interval $[a,+\infty[$, we consider the problem on the existence of a solution $(u_1,u_2):[a,+\infty[\,\to\mathbb{R}^2$ of the differential system

$$u_1^{(m)}=f_1(t,u_2),\quad u_2^{(m)}=f_2(t,u_1), \tag{1}$$

satisfying the boundary conditions

$$\varphi_k\big(u_1^{(i-1)}(a),u_2^{(m-i)}(a)\big)=0\ \ (k=1,\dots,m),\quad \Phi_m(u_1,u_2)<+\infty, \tag{2}$$

where $f_i:[a,+\infty[\,\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ and $\varphi_k:\mathbb{R}^2\to\mathbb{R}$ $(k=1,\dots,m)$ are continuous functions and

$$\Phi_m(u_1,u_2)=\int\limits_a^{+\infty}\Big(\big|u_1^{(m)}(t)u_2(t)\big|+\big|u_2^{(m)}(t)u_1(t)\big|\Big)\,dt.$$

Analogous problems arise in the oscillation theory and in the case, where $f_1(t,x)\equiv x$, they have been studied in [1,2]. In a general case, problem (1), (2) remains still unstudied. The results established by us fill to a certain extent the existing here gap.

We study the case in which the functions f_i $(i=1,2)$ on the set $[a,+\infty[\,\times\mathbb{R}$ satisfy the inequalities

$$f_1(t,x)x\geq g_1(t,x),\quad (-1)^{m-1}f_2(t,x)x\geq g_2(t,x), \tag{3}$$

where $g_i:[a,+\infty[\,\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are continuous functions such that

$$\begin{gathered} g_i(t,x)\leq g_i(t,y)\ \text{ for }\ t\geq a,\ \ xy\geq 0,\ \ |x|<|y|\ \ (i=1,2),\\ \int\limits_a^b|g_i(t,0)|\,dt<+\infty\ \ (i=1,2). \end{gathered} \tag{4}$$

Along with (1), (2), we consider the auxiliary problem

$$u_1^{(m)}=(1-\lambda)u_2+\lambda f_1(t,u_2),\quad u_2^{(m)}=(-1)^{m-1}(1-\lambda)u_1+\lambda f_2(t,u_1), \tag{5}$$

$$(1-\lambda)u_1^{(k-1)}(a)+\lambda\varphi\big(u_1^{(k-1)}(a),u_2^{(m-k)}(a)\big),\quad u_1^{(k-1)}(b)=0\ \ (k=1,\dots,m), \tag{6}$$

depending on the parameters $\lambda\in[0,1]$ and $b\in\,]a,+\infty[$.

Proposition 1 (The principle of a priori boundedness)**.** *Let there exist constants $b_0>a$ and $r>0$ such that for any $\lambda\in[0,1]$ and $b\geq b_0$ every solution of problem* (5), (6) *admits the estimate*

$$\sum_{k=1}^m\big(|u_1^{(k-1)}(a)|+|u_2^{(k-1)}(a)|\big)+\Phi_m(u_1,u_2)\leq r.$$

Then problem (1), (2) *has at least one solution.*

Based on Proposition 1, the following theorem can be proved.

Theorem 1. *Let conditions* (3), (4) *be fulfilled, there exist mutually nonintersecting intervals* $[a_{ik}, b_{ik}] \subset [a, +\infty[$ $(i = 1, 2;\ k = 1, \dots, m)$ *and a positive number* r *such that*

$$\lim_{|x|\to+\infty} \int\limits_{a_{ik}}^{b_{ik}} g_i(t,x)\,dt = +\infty \ \ (i = 1, 2;\ \ k = 1, \dots, m), \tag{7}$$

$$\varphi_k(x,y)x > 0 \ \text{ for } \ (-1)^{m-k+1}xy > r \ \ (k = 1, \dots, m).$$

Then problem (1), (2) *has at least one solution.*

Particular cases of (2) are the boundary conditions

$$\begin{gathered} u_1^{(j_k-1)}(a) = \psi_k(u_2^{(m-j_k)}(a)) \ \ (k = 1, \dots, m_0), \\ u_2^{(m-j_k)}(a) = \psi_k(u_1^{(j_k-1)}(a)) \ \ (k = m_0+1 \dots, m), \quad \Phi_m(u_1, u_2) < +\infty; \end{gathered} \tag{8}$$

$$u_1^{(k-1)}(a) = \psi_k(u_2^{(m-k)}(a)) \ \ (k = 1, \dots, m), \quad \Phi_m(u_1, u_2) < +\infty, \tag{9}$$

and

$$u_2^{(m-k)}(a) = \psi_k(u_1^{(k-1)}(a)) \ \ (k = 1, \dots, m), \quad \Phi_m(u_1, u_2) < +\infty, \tag{10}$$

where

$$m_0 \in \{1, \dots, m-1\}, \ \ j_k \in \{1, \dots, m\}, \ \ j_k \neq j_\ell \ \text{ for } \ k \neq \ell,$$

and $\psi_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, m)$ are continuous functions.

Corollary 1. *If along with* (3), (4) *and* (7) *the condition*

$$\liminf_{|x|\to+\infty} \big[(-1)^{m-j_k}\psi_k(x)x\big] > -\infty \tag{11}$$

is fulfilled, then problem (1), (8) *has at least one solution.*

Corollary 2. *If along with* (3), (4) *and* (7) *the condition*

$$\liminf_{|x|\to+\infty} \big[(-1)^{m-k}\psi_i(x)x\big] > -\infty \ \ (k = 1, \dots, m) \tag{12}$$

is fulfilled, then problem (1), (9), *as well as problem* (1), (10) *has at least one solution.*

An interesting particular case of system (1) is the Emden–Fowler type differential system

$$u_1^{(m)} = p_1(t)|u_2|^{\lambda_1}\operatorname{sgn}(u_2) + q_1(t), \quad u_2^{(m)} = p_2(t)|u_1|^{\lambda_2}\operatorname{sgn}(u_1) + q_2(t), \tag{13}$$

where $\lambda_i > 0$ $(i = 1, 2)$, and $p_i : [a, +\infty[\to \mathbb{R}$, $q_i : [a, +\infty[\to \mathbb{R}$ $(i = 1, 2)$ are continuous functions.

Corollary 3. *Let*

$$p_1(t) \geq 0, \quad (-1)^{m-1}p_2(t) \geq 0 \ \text{ for } \ t \geq a,$$

$$|q_i(t)| \leq q_{i0}(t)|p_i(t)|^{\frac{1}{1+\lambda_i}} \ \text{ for } \ t \geq a \ \ (i = 1, 2),$$

where $q_{i0} : [a, +\infty[\to [0, +\infty[$ $(i = 1, 2)$ *are continuous functions such that*

$$\int\limits_a^{+\infty} |q_{i0}(t)|^{1+\frac{1}{\lambda_i}}\,dt < +\infty \ \ (i = 1, 2).$$

If, moreover, condition (12) *holds, then problem* (13), (9), *as well as problem* (13), (10) *has at least one solution.*

Consider now the case, where

$$\begin{gathered} f_i(t,0)=0 \ \text{ for } \ t\ge a \ \ (i=1,2),\\ f_1(t,x)\le f_1(t,y), \quad (-1)^{m-1}f_2(t,x)\le(-1)^{m-1}f_2(t,y) \ \text{ for } \ t\ge a, \ \ x<y. \end{gathered} \tag{14}$$

A nontrivial solution (u_1,u_2) of system (1) defined on some infinite interval $[a_0,+\infty[\,\subset[a,+\infty[$ is called:

(i) **proper** if it is not identically equal to zero in any neighbourhood of $+\infty$;

(ii) **oscillatory** if every its component changes sign in any neighbourhood of $+\infty$;

(iii) **first kind singular solution** if there exists $t_0>a_0$ such that $u_i(t)=0$ for $t\ge t_0$ $(i=1,2)$.

The problem on the existence of proper (in particular, oscillatory) solutions of system (1) is of special interest in the so-called "superlinear" case, when the right-hand sides of that system are the functions, rapidly growing with respect to the phase variables. Such are the cases that cover the results formulated below.

Corollary 4. *Let along with* (12) *and* (14) *the conditions*

$$f_i(t_i,x_0)\neq0 \ \ (i=1,2), \quad \sum_{k=1}^m|\psi_k(0)|>0$$

be fulfilled, where $t_i>a$ $(k=1,2)$ and $x_0\neq0$. If, moreover, system (1) *has no first kind singular solution, then problems* (1), (9) *and* (1), (10) *are solvable, and their solutions are proper.*

Theorem 2. *Let system* (1) *have no first kind singular solution and along with* (12), (14) *the condition*

$$\int_a^{+\infty}|f_i(t,x)|\,dt=+\infty \ \ (i=1,2)$$

be fulfilled. Let, moreover, either

$$m \ \textit{be even and} \ \sum_{k=1}^m|\psi_k(0)|>0,$$

or

$$m\ge3 \ \textit{be odd}, \ \ \psi_m(x)\equiv0, \ \ \sum_{k=1}^{m-1}|\psi_k(0)|>0.$$

Then problems (1), (9) *and* (1), (10) *are solvable, and their solutions are oscillatory.*

As an example, we consider the problem

$$u_1^{(m)}=p_1(t)\exp(h_1(t)|u_2|)|u_2|^{\lambda_1}\operatorname{sgn}(u_2), \quad u_2^{(m)}=p_2(t)\exp(h_2(t)|u_1|)|u_1|^{\lambda_2}\operatorname{sgn}(u_1), \tag{15}$$

$$u_1^{(k-1)}(a)=\alpha_k u^{(m-k)}(a)+c_k \ \ (k=1,\dots,m), \quad I_m(u_1,u_2)<+\infty, \tag{16}$$

where

$$\lambda_1>0, \ \ \lambda_1\lambda_2\ge1, \ \ \alpha_k\in\mathbb{R} \ \ (i=1,\dots,m),$$

and $p_i,h_i:[a,+\infty[\,\to\mathbb{R}$ $(i=1,2)$ are continuous functions such that

$$p_1(t)\ge0, \quad (-1)^{m-1}p_2(t)\ge0, \quad h_1(t)\ge0, \ \ h_2(t)\ge0 \ \text{ for } \ t\ge a.$$

From Theorem 2 it follows

Corollary 5. *Let*

$$\int\limits_a^{+\infty} p_i(t)\exp(xh_i(t))\,dt = +\infty \ \text{ for } \ x>0 \ \ (i=1,2)$$

and let, moreover, either

$$m \text{ be even}, \ \ (-1)^{m-k}\alpha_k>0 \ \ (k=1,\dots,m), \ \ \sum_{k=1}^{m}|c_k|>0,$$

or

$$m \text{ be odd}, \ \ (-1)^{m-k}\alpha_k>0 \ \ (k=1,\dots,m-1), \ \ \alpha_m=c_m=0, \ \ \sum_{k=1}^{m-1}|c_k|>0.$$

Then problem (15), (16) *is solvable and every its solution is oscillatory.*

References

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