The Neumann problem for two-term fractional differential equations

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1. Formulation of problem

Let $T > 0$ and $J = [0, T]$.

We consider the Neumann problem

$$^cD^\alpha u(t) = a(t)\,^cD^\beta u(t) + f(t, u(t)), \tag{1}$$

$$u'(0) = 0, \quad u'(T) = 0, \tag{2}$$

where $\alpha \in (1, 2)$, $\beta \in (\alpha - 1, \alpha)$, $a \in C(J)$, $f \in C(J \times \mathbb{R})$, ^cD denotes the Caputo fractional derivative.

We say that a function $u : J \to \mathbb{R}$ is a solution of the problem (1), (2) if $u, {}^cD^\alpha u \in C(J)$, u satisfies (2) and (1) holds for $t \in J$.

Since any constant function u on J is a solution of the problem $^cD^\alpha u = a(t)\,^cD^\beta u$, $u'(0) = u'(T) = 0$, problem (1), (2) is at resonance.

In order to give a solution of problem (1), (2), we define a two-component integral operator $\mathcal{Q} : C(J) \times \mathbb{R} \times [0,1] \to C(J) \times \mathbb{R}$ and prove that if (x, c) is a fixed point of $\mathcal{Q}(\cdot, \cdot, 1)$, then x is a solution of (1), (2). The existence of a fixed point is proved by the Leray-Schauder degree method.

2. Fractional calculus

The Riemann-Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function.

- $I^\gamma I^\mu x(t) = I^{\gamma+\mu} x(t)$ for $t \in J$, $x \in C(J)$, $\gamma, \mu \in (0, \infty)$ - semigroup property
- $I^\gamma : C(J) \to C^{n-1}(J)$ for $\gamma \in (n-1, n)$, $n \in \mathbb{N}$

The Caputo fractional derivative ${}^cD^\gamma x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$$ {}^cD^\gamma x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) \mathrm{d}s, $$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of γ.
${}^cD^\gamma x(t) = x^{(\gamma)}(t)$ for $\gamma \in \mathbb{N}$.

In particular,

$$ {}^cD^\gamma x(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t \frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} (x(s) - x(0) - x'(0)s) \, \mathrm{d}s, \quad \gamma \in (1,2), $$

and if $x \in C^2(J)$, then

$$ {}^cD^\gamma x(t) = \int_0^t \frac{(t-s)^{1-\gamma}}{\Gamma(2-\gamma)} x''(s) \, \mathrm{d}s = I^{2-\gamma} x''(t), \quad t \in J, \ \gamma \in (1,2). $$

- ${}^cD^\gamma I^\gamma x(t) = x(t)$ for $t \in J$, $x \in C(J)$, $\gamma > 0$
- $I^\gamma {}^cD^\gamma x(t) = x(t) - x(0) - x'(0)t$ for $t \in J$, $x, {}^cD^\gamma x \in C(J)$, $\gamma \in (1,2)$.

We can write equation (1) as ($\alpha \in (1,2)$, $\beta \in (\alpha-1,\alpha)$)

$\beta \in (1,\alpha)$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}u(s)\,\mathrm{d}s = a(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}u(s)\,\mathrm{d}s + f(t,u(t))$$

$\beta = 1$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}u(s)\,\mathrm{d}s = a(t)u'(t) + f(t,u(t))$$

$\beta \in (\alpha-1,1)$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}u(s)\,\mathrm{d}s = a(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}u(s)\,\mathrm{d}s + f(t,u(t))$$

J.R. Graef, L. Kong, Q. Kong, M. Wang, *Positive solutions of nonlocal fractional boundary value problems*, Discrete Contin. Dyn. Syst., supl. 2013, 283–290

The problem

$$-D^{\alpha}u(t) + aD^{\gamma}u(t) = f(t, u(t)),$$

$$D^{\beta}u(t)|_{t=0} = 0, \quad D^{\alpha-\gamma}u(t)|_{t=1} = au(1)$$

is discussed, vhere D^{α} is the Riemann-Liouville fractional derivative, $1 < \gamma < \alpha \le 2$, $0 \le \beta < \alpha - \gamma$, $0 \le a < \Gamma(\alpha - \gamma + 1)$.

The existence of a positive solution is proved by using the Green function and fixed point theory on cones.

3. Preliminaries

Let $\Lambda : C(J) \to C(J)$ be defined by

$$\Lambda x(t) = a(t) I^{\alpha-\beta} x(t),$$

where $a(t)$ is from (1), and let Λ^n, $n \in \mathbb{N}$, be the n-th iteration of Λ, that is,

$$\Lambda^n = \underbrace{\Lambda \circ \Lambda \circ \cdots \circ \Lambda}_{n}.$$

Let $\mathcal{A} : C(J) \to C(J)$ be given as

$$\mathcal{A} x(t) = \sum_{n=1}^{\infty} \Lambda^n x(t).$$

For $\gamma > 0$ let E_γ be the classical Mittag–Leffler function acting on $\mathbb{R}$,

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)}.$$

$$|\mathcal{A} x(t)| \le \|x\| \left(E_{\alpha-\beta} \left(\|a\| t^{\alpha-\beta} \right) - 1 \right), \quad t \in J,\ x \in C(J).$$

LEMMA 1. *Let* $r \in C(J)$ *and* $c \in \mathbb{R}$. *Then the fuction*

$$u(t) = c + I^{\alpha} r(t) + I^{\alpha} \mathcal{A} r(t), \quad t \in J,$$

is the unique solution of the initial fractional value problem

$$\left. \begin{array}{c} {}^{c}D^{\alpha} u(t) = a(t)\, {}^{c}D^{\beta} u(t) + r(t), \\ u(0) = c, \quad u'(0) = 0. \end{array} \right\}$$

We will show how to get the integral representation

$$u(t) = c + I^\alpha r(t) + I^\alpha \mathcal{A} r(t), \quad t \in J,$$

of IVP

$${}^cD^\alpha u = a(t){}^cD^\beta u + r(t), \quad u(0) = c, \ u'(0) = 0.$$

Suppose that u is a solution of ${}^cD^\alpha u = a(t){}^cD^\beta u + r(t)$ satisfying $u'(0) = 0$. Since ${}^cD^\beta u = I^{\alpha-\beta}\,{}^cD^\alpha u$, we have

$${}^cD^\alpha u = a(t) I^{\alpha-\beta}\,{}^cD^\alpha u + r(t).$$

Let $z = {}^cD^\alpha u$. Then $z = a(t)I^{\alpha-\beta} z + r(t)$, that is,

$$z = \Lambda z + r(t).$$

Then

$$z = \Lambda(\Lambda z + r(t)) + r(t) = \Lambda^2 z + \Lambda r(t) + r(t),$$

$$z = \Lambda^{n+1} z + r(t) + \Lambda r(t) + \cdots \Lambda^n r(t) = \Lambda^{n+1} z + r(t) + \sum_{k=1}^{n} \Lambda^k r(t).$$

Letting $n \to \infty$ and supposing $\lim_{n\to\infty} \Lambda^n z = 0$, we arrive at

$$z = r(t) + \sum_{k=1}^{\infty} \Lambda^k r(t) = r(t) + \mathcal{A}r(t)$$

Hence

$$^{c}D^{\alpha} u = r(t) + \mathcal{A}r(t)$$

By applying I^{α} to the last equality and using $u(0) = c$, we get

$$u(t) = c + I^{\alpha} r(t) + I^{\alpha} \mathcal{A}r(t), \quad t \in J$$

4. Operators

Keeping in mind Lemma 1, let $\mathcal{F} : C(J) \to C(J)$ be defined as

$$(\mathcal{F}x)(t) = I^{\alpha} f(t, x(t)) + I^{\alpha} \mathcal{A} f(t, x(t))$$

and let $\mathcal{Q} : C(J) \times \mathbb{R} \times [0, 1] \to C(J) \times \mathbb{R}$,

$$\mathcal{Q}(x, c, \lambda) = \Big(c + \lambda(\mathcal{F}x)(t),\ c + I^{\alpha-1} f(t, x(t))|_{t=T} + I^{\alpha-1} \mathcal{A} f(t, x(t))|_{t=T} \Big).$$

LEMMA 2. *$\mathcal{Q}$ is a completely continuous operator and if (x, c) is a fixed point of $\mathcal{Q}(\cdot, \cdot, 1)$, then x is a solution of problem* (1), (2) *and $c = x(0)$.*

We will work with the following conditions on the functions a and f in equation (1):

(H_1) $a \in C(J)$ and $a(t) \geq 0$ for $t \in J$.

(H_2) $f \in C(J \times \mathbb{R})$ and there exists positive constant S such that

$$xf(t,x) > 0 \quad \text{for } t \in J \text{ and } |x| \geq S.$$

(H_3) There exist positive constants K and L such that

$$|f(t,x)| \leq K + L|x| \quad \text{for } (t,x) \in J \times \mathbb{R}.$$

LEMMA 3. *Let $(H_1) - (H_3)$ hold. Then there exists a positive constant W such that the estimate*

$$\|x\| < W, \quad |c| < W,$$

is fulfilled for all fixed points (x,c) of the operator $\mathcal{Q}(\cdot,\cdot,\lambda)$ with $\lambda \in [0,1]$.

5. Existence and uniqueness

THEOREM 1. *Let* $(H_1) - (H_3)$ *hold. Then the Neumann problem* (1), (2) *has at least one solution.*

Proof. Let $W > 0$ be from Lemma 3 and $\Omega = \{x \in C(J)\colon \|x\| \le W\}$. By the Borsuk antipodal theorem and the homotopy property,

$$\deg(\mathcal{I} - \mathcal{Q}(\cdot,\cdot,0), \Omega, 0) \neq 0,$$

$$\deg(\mathcal{I} - \mathcal{Q}(\cdot,\cdot,0), \Omega, 0) = \deg(\mathcal{I} - \mathcal{Q}(\cdot,\cdot,1), \Omega, 0).$$

where $\mathcal{I}$ is the identical operator on $C(J) \times \mathbb{R}$. Hence

$$\deg(\mathcal{I} - \mathcal{Q}(\cdot,\cdot,1), \Omega, 0) \neq 0,$$

and therefore there exists a fixed point $(u, c) \in \Omega$ of the operator $\mathcal{Q}(\cdot,\cdot,1)$. By Lemma 2, u is a solution of problem (1), (2).

EXAMPLE 1. The fractional differential equation

$$ {}^{c}D^{19/10}u = |\sin t|\,{}^{c}D^{18/10}u + t\cos u + \frac{u^2 \arctan u}{1+|u|} \tag{3} $$

satisfies conditions $(H_1)-(H_3)$ for $\alpha = 19/10$, $\beta = 18/10$, $S = \max\{2T, 1\}$, $K = T$ and $L = \pi/2$. Hence the Neumann problem (3), (2) has at least one solution.

THEOREM 2. *Let* $(H_1)-(H_3)$ *hold. Let* $f(t,x)$ *be increasing in the variable* x *for all* $t \in J$*, and for each* $M > 0$ *there exist* $L_M > 0$ *such that*

$$|f(t,x) - f(t,y)| \le L_M|x-y| \quad \text{for } t \in J,\ x,y \in [-M,M].$$

Then problem (1), (2) *has a unique solution.*

EXAMPLE 2. Let $b \in C(J)$ and $f(t,x) = b(t) + x^3/(1+x^2)$ Then f satisfies conditions (H_2) and (H_3) for $S = 2\max\{1, \|b\|\}$, $K = \|b\|$ and $L = 1$. Besides, $f(t,\cdot)$ is increasing for all $t \in J$ and

$$|f(t,x) - f(t,y)| \le 3|x-y|, \quad t \in J,\ x,y \in \mathbb{R}.$$

According to Theorem 2, for each a satisfying (H_1) the problem

$$\left.\begin{array}{c} {}^cD^\alpha u = a(t)\,{}^cD^\beta u + b(t) + \dfrac{u^3}{1+u^2}, \\ u'(0) = 0, \quad u'(T) = 0 \end{array}\right\}$$

has a unique solution.

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- K. Diethelm, *The Analysis of Fractional Differential Equations*, Lectures Notes in Mathematics, Springer Berlin Heidelberg 2010.