## Non-negative periodic solutions of second-order differential equations with sublinear nonlinearities

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$$u'' = p(t)u + q(t,u); \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

- $p \colon [0,\omega] \to \mathbb{R}$  ... Lebesgue integrable
- $q: [0, \omega] \times \mathbb{R} \to \mathbb{R} \dots$  Carathéodory + sublinear

(\*)

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- $\triangleright$  solution =  $AC^1$  function

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$$\begin{aligned} |q(t,x)| &\leq q_0(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq x_0, \\ x_0 &> 0, \quad q_0 \colon [0,\omega] \times [x_0, +\infty[ \to [0, +\infty[ \text{ is a Carathéodory function}, \\ q_0(t,\cdot) \colon [x_0, +\infty[ \to [0, +\infty[ \text{ is non-decreasing for a. e. } t \in [0,\omega], \\ \lim_{x \to +\infty} \frac{1}{x} \int_0^{\omega} q_0(s,x) \mathrm{d}s = 0. \end{aligned} \right\}$$
(H1)

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$$\begin{array}{l} q(t,x) \geq xg(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in ]0,\delta], \\ \delta > 0, \quad g \colon [0,\omega] \times ]0,\delta] \to \mathbb{R} \text{ is a locally Carathéodory function,} \\ g(t,\cdot) \colon ]0,\delta] \to \mathbb{R} \text{ is non-increasing for a. e. } t \in [0,\omega], \end{array} \right\}$$

$$\left. \begin{array}{l} (H_2) \\ (H_2) \\ (H_3) \\ (H$$

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$$\left. \begin{array}{l} \text{for every } b > a > 0, \text{ there exists } h_{ab} \in L([0, \omega]) \text{ such that} \\ h_{ab}(t) \ge 0 \quad \text{for a. e. } t \in [0, \omega], \quad h_{ab} \not\equiv 0, \\ q(t, x) \ge h_{ab}(t) \quad \text{for a. e. } t \in [0, \omega] \text{ and all } x \in [a, b], \end{array} \right\}$$

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- we are interested in the existence and uniqueness of non-trivial non-negative as well as positive solutions of (\*)

$$\begin{array}{l} \text{For every } b > a > 0 \ \text{and } c > 0, \ \text{there exists } h_{abc} \in L([0,\omega]) \ \text{such that} \\ h_{abc}(t) \ge 0 \quad \text{for a. e. } t \in [0,\omega], \ h_{abc} \not\equiv 0, \\ \frac{q(t,x)}{x} - \frac{q(t,x+c)}{x+c} \ge h_{abc}(t) \quad \text{for a. e. } t \in [0,\omega] \ \text{and all } x \in [a,b]. \end{array} \right\}$$

$$u'' = p(t)u + q(t, u); \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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- $\triangleright$  solution =  $AC^1$  function
- we are interested in the existence and uniqueness of non-trivial non-negative as well as positive solutions of (\*)
- ▷ particular case:

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

*p*, *h* ∈ *L*([0, ω])
λ ∈ ]0, 1[

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λ ∈ ]0, 1[

either 
$$h \ge 0$$
 a.e. on  $[0, \omega]$ , or  $h \le 0$  a.e. on  $[0, \omega]$ ,

the coefficient p is not constant and can change its sign!!!

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

• We say that  $\left| \begin{array}{c} p \in \mathcal{V}^+(\omega) \end{array} 
ight|$  if

$$egin{array}{lll} u\in AC^1([0,\omega]),\ u''(t)\geq p(t)u(t) & ext{for a. e. }t\in [0,\omega],\ u(0)=u(\omega), \ u'(0)=u'(\omega) \end{array}
ight\} \implies u(t)\geq 0 & ext{for }t\in [0,\omega]. \end{array}$$

Alternatively - Green's function is positive, or antimaximum principle holds

• We say that  $p \in \mathcal{V}^{-}(\omega)$  if  $u \in AC^{1}([0, \omega]),$   $u''(t) \ge p(t)u(t)$  for a.e.  $t \in [0, \omega],$   $u(0) = u(\omega), u'(0) = u'(\omega)$  $w(t) \le 0$  for  $t \in [0, \omega].$ 

Alternatively - Green's function is negative, or maximum principle holds

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

• We say that  $\left| \ p \in \mathcal{V}_0(\omega) 
ight|$  if the problem

$$u''=p(t)u; \quad u(0)=u(\omega), \,\, u'(0)=u'(\omega)$$

has a positive solution.

• We say that  $p \in D_1(\omega)$  if for any  $\alpha \in [0, \omega[$ , the solution u of the problem the problem

$$u^{\prime\prime}=\widetilde{p}(t)u; \hspace{0.4cm} u(lpha)=0, \hspace{0.4cm} u^{\prime}(lpha)=1$$

has at most one zero on the interval  $]\alpha, \alpha + \omega[$ , where  $\tilde{p}$  is the  $\omega$ -periodic extension of p to the whole real axis.

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$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

•  $h(t) \geq 0$  for a.e.  $t \in [0, \omega], \ h \not\equiv 0$ 

$$u^{\prime\prime} = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u^{\prime}(0) = u^{\prime}(\omega)$$
 (1)

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•  $h(t) \geq 0$  for a.e.  $t \in [0, \omega]$ ,  $h \not\equiv 0$ 

$$y^{\prime\prime}=ay+b\sqrt[3]{y}$$

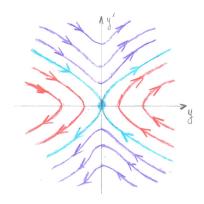
• b > 0

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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•  $h(t) \geq 0$  for a.e.  $t \in [0, \omega]$ ,  $h \not\equiv 0$ 

$$y^{\prime\prime}=ay+b\sqrt[3]{y}$$

• b > 0,  $a \ge 0$ 

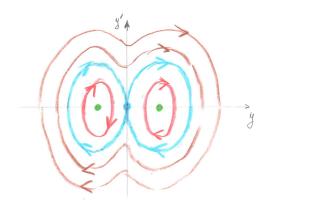


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•  $h(t) \geq 0$  for a.e.  $t \in [0, \omega], \ h \not\equiv 0$ 

$$y^{\prime\prime}=ay+b\sqrt[3]{y}$$

• b > 0, a < 0



$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

$$h(t) > 0$$
 for a.e.  $t \in [0, \omega]$ . (A<sub>1</sub>)

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Then:

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega) \,\, \left( 1 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{2} \int$$

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)

Then:

(1)  $p \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has only the trivial solution

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Then:

- (1)  $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \implies$  (1) has only the trivial solution
- (2)  $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions  $(\underset{\neq}{\geq} 0, \underset{\neq}{\leq} 0, \equiv 0)$

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**Example.** Consider a particular case of (1) with

$$p(t):=-1, \quad h(t):=3(1-\sin t) \quad ext{for } t\in [0,2\pi], \qquad \lambda:=rac{1}{2} \ , \qquad \omega:=2\pi,$$

namely, the problem

$$u'' = -u + 3(1 - \sin t)\sqrt{|u|} \operatorname{sgn} u; \quad u(0) = u(2\pi), \ u'(0) = u'(2\pi).$$
 (2)

Then  $p \in \mathcal{D}_1(\omega)$ ,  $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)$ , (A<sub>1</sub>) holds, and problem (2) has a solution

$$u(t):=\left(1+\sin t
ight)^2$$
 for  $t\in[0,2\pi]$  .

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

$$h(t) > 0$$
 for a.e.  $t \in [0, \omega]$ .  $(A_1)$ 

Then:

p ∈ V<sup>-</sup>(ω) ∪ V<sub>0</sub>(ω) ⇒ (1) has only the trivial solution
 p ∉ V<sup>-</sup>(ω) ∪ V<sub>0</sub>(ω) ⇒ (1) has at least 3 sign-constant solutions (<sup>≥</sup><sub>₹</sub>0, <sup>≤</sup><sub>₹</sub>0, ≡ 0)
 (2a) p ∈ V<sup>+</sup>(ω) ⇒ (1) has exactly 3 solutions (> 0, < 0, ≡ 0)</li>

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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$$h(t) > 0$$
 for a.e.  $t \in [0, \omega]$ . (A<sub>1</sub>)

Then:

(1)  $p \in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has only the trivial solution (2)  $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions  $(\stackrel{>}{\neq} 0, \stackrel{\leq}{\neq} 0, \equiv 0)$ (2a)  $p \in \mathcal{V}^{+}(\omega) \Rightarrow$  (1) has exactly 3 solutions (> 0, < 0,  $\equiv 0$ ) (2b)  $p \in \mathcal{D}_{1}(\omega) \setminus [\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \mathcal{V}^{+}(\omega)] \Rightarrow$  (1) has at least 3 sign-constant solutions and no sign-changing solutions

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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$$h(t) > 0$$
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Then:

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 (1) has only the trivial solution  
(2)  $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions  $(\stackrel{\geq}{\neq} 0, \stackrel{\leq}{\neq} 0, \equiv 0)$   
(2a)  $p \in \mathcal{V}^{+}(\omega) \Rightarrow$  (1) has exactly 3 solutions (> 0, < 0,  $\equiv 0$ )  
(2b)  $p \in \mathcal{D}_{1}(\omega) \setminus [\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \mathcal{V}^{+}(\omega)] \Rightarrow$  (1) has at least 3 sign-constant  
solutions and no sign-changing solutions  
(2c)  $p \notin \mathcal{D}_{1}(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions

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 for a.e.  $t \in [0, \omega]$ . (A<sub>1</sub>)

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(2b)  $p \in \mathcal{D}_{1}(\omega) \setminus [\mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \mathcal{V}^{+}(\omega)] \Rightarrow$  (1) has at least 3 sign-constant  
solutions and no sign-changing solutions  
(2c)  $p \notin \mathcal{D}_{1}(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions

**Remark:** Assertions (1) and (2a) remain true even if  $(A_1)$  is relaxed to

$$h(t) \ge 0$$
 for a.e.  $t \in [0, \omega], \quad h \not\equiv 0.$  (A<sub>2</sub>)

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$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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$$h(t) > 0$$
 for a.e.  $t \in [0, \omega]$ . (A<sub>1</sub>)

Then:

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solutions and no sign-changing solutions

(2c)  $p \notin \mathcal{D}_1(\omega) \Rightarrow$  (1) has at least 3 sign-constant solutions

**Remark:** Assertions (1) and (2a) remain true even if  $(A_1)$  is relaxed to

$$h(t) \geq 0$$
 for a.e.  $t \in [0, \omega],$   $h \not\equiv 0.$   $(A_2)$ 

## **Open questions:**

- $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \cup \mathcal{V}^{+}(\omega) \implies (1)$  has a positive solution?
- $p \notin \mathcal{D}_1(\omega) \implies (1)$  has a sign-changing solution?

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)$$

(1)

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(1)

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 $\triangleright \ p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \implies (1) \text{ has at least one non-trivial non-negative solution}$ •  $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \implies \exists \ell \in L([0, \omega]), \text{ such that } \ell \geq 0 \text{ and } p + \ell \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ 

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega) \,\, \Big| \,\,$$

 $rac{>} p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow$  (1) has at least one non-trivial non-negative solution

- $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow \exists \ell \in L([0, \omega]), \text{ such that } \ell \geq 0 \text{ and } p + \ell \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ 
  - $\Rightarrow$   $\exists$  an arbitrarily large positive lower function  $\alpha$  of problem (1)

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)$$

 $ightarrow p 
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- $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \Rightarrow \exists \ell \in L([0, \omega])$ , such that  $\ell \geq 0$  and  $p + \ell \in \text{Int } \mathcal{V}^{+}(\omega)$  $\Rightarrow \exists$  an arbitrarily large positive lower function  $\alpha$  of problem (1)
- $\exists r > 0$  such that  $p + rac{h}{r^{1-\lambda}} \in \mathcal{V}^-(\omega)$

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)\,\,\Big|$$

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ot\in \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \;\;\Rightarrow\;\;$  (1) has at least one non-trivial non-negative solution

- $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \implies \exists \ell \in L([0, \omega])$ , such that  $\ell \geq 0$  and  $p + \ell \in \text{Int } \mathcal{V}^{+}(\omega)$  $\Rightarrow \exists$  an arbitrarily large positive lower function  $\alpha$  of problem (1)
- $\exists r > 0$  such that  $p + \frac{h}{r^{1-\lambda}} \in \mathcal{V}^{-}(\omega) \implies \exists$  an arbitrarily small positive upper function  $\beta$  of problem (1)

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)\,\,\Big|$$

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- $\exists r > 0$  such that  $p + \frac{h}{r^{1-\lambda}} \in \mathcal{V}^{-}(\omega) \implies \exists$  an arbitrarily small positive upper function  $\beta$  of problem (1)
- $\delta > 0$  large enough, cutting function  $\chi(x) := [x]_+ [x-\delta]_+$  for  $x \in \mathbb{R}$

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega) \,\, \Big| \,\,$$

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- $\exists r > 0$  such that  $p + \frac{h}{r^{1-\lambda}} \in \mathcal{V}^{-}(\omega) \implies \exists$  an arbitrarily small positive upper function  $\beta$  of problem (1)
- $\delta > 0$  large enough, cutting function  $\chi(x) := [x]_+ [x-\delta]_+$  for  $x \in \mathbb{R}$
- auxiliary problem

$$u'' = (p(t) + \ell(t))u + h(t)|\chi(u)|^{\lambda} \operatorname{sgn} \chi(u) - \ell(t)\chi(u); \quad \mathsf{PBC}$$
 (3)

$$u'' = p(t)u + h(t)|u|^{\lambda} \, {
m sgn} \, u; \quad u(0) = u(\omega), \, \, u'(0) = u'(\omega) \, \, igg|$$

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•  $(\alpha, \beta)$  is a couple of reverse-ordered lower and upper functions of (3)

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 PBC (3)

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- (3) has a solution u such that

$$0 < \beta(t_u) \le u(t_u) \le \alpha(t_u)$$
 for some  $t_u \in [0, \omega]$  (4)

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- $p + \ell \in \operatorname{Int} \mathcal{V}^+(\omega) \quad \Rightarrow \quad u(t) \geq 0 \, \, ext{for} \, t \in [0, \omega]$
- (5)  $\Rightarrow$   $u \not\equiv 0$

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- $\bullet \ p+\ell \in \operatorname{Int} \mathcal{V}^+(\omega) \ \ \Rightarrow \ \ u(t) \geq 0 \ \text{for} \ t \in [0,\omega]$
- (5)  $\Rightarrow$   $u \not\equiv 0$
- $u(t) \le \delta$  for  $t \in [0, \omega] \Rightarrow \chi(u) \equiv u \Rightarrow u$  is a non-trivial non-negative solution of (1)

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)$$

 $arproptop p 
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 $\vartriangleright p \in {\mathcal D}_1(\omega) \text{ and } u \text{ is a solution of } (1) \hspace{0.1 cm} \Rightarrow \hspace{0.1 cm}$ 

either  $u(t) \geq 0$  for  $t \in [0, \omega]$ , or  $u(t) \leq 0$  for  $t \in [0, \omega]$ 



(1)

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)$$

 $ightarrow p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_{0}(\omega) \implies$  (1) has at least one non-trivial non-negative solution  $ightarrow p \in \mathcal{D}_{1}(\omega)$  and u is a solution of (1)  $\Rightarrow$ either  $u(t) \ge 0$  for  $t \in [0, \omega]$ , or  $u(t) \le 0$  for  $t \in [0, \omega]$ 

 $dash p \in \mathcal{V}^+(\omega)$  and u is a solution of (1)  $\Rightarrow$ either u(t) > 0 for  $t \in [0, \omega]$ , or  $u(t) \le 0$  for  $t \in [0, \omega]$ 

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

•  $h(t) \leq 0$  for a.e.  $t \in [0, \omega], \ h \not\equiv 0$ 



$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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•  $h(t) \leq 0$  for a.e.  $t \in [0, \omega], \ h \not\equiv 0$ 

$$y^{\prime\prime}=ay+b\sqrt[3]{y}$$

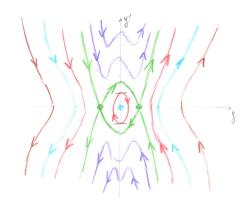
● *b* < 0

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)$$

•  $h(t) \leq 0$  for a.e.  $t \in [0, \omega], \ h \not\equiv 0$ 

$$y^{\prime\prime}=ay+b\sqrt[3]{y}$$

• b < 0, a > 0



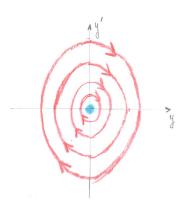
(1)

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• b < 0,  $a \le 0$ 



(1)

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
 (1)

**Theorem.** Let  $\lambda \in ]0, 1[$  and

$$h(t) \leq 0$$
 for a.e.  $t \in [0, \omega], \qquad h \not\equiv 0.$  (A<sub>3</sub>)

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Then:

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega)$$
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Then:

(1)  $p \in \mathcal{V}^{-}(\omega) \Rightarrow$  (1) has exactly 3 solutions (> 0, < 0,  $\equiv$  0)

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**Theorem.** Let  $\lambda \in ]0, 1[$  and

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### **Open questions:**

- $p \notin \mathcal{V}^{-}(\omega) \implies (1)$  has a non-trivial non-negative solution?
- $p \notin \mathcal{V}^{-}(\omega) \implies (1)$  has a sign-changing solution?

$$u^{\prime\prime} = p(t) u + h(t) |u|^{\lambda} \, {
m sgn} \, u; \quad u(0) = u(\omega), \, \, u^{\prime}(0) = u^{\prime}(\omega) \, \, igg|$$

 $arphi \ p \in \mathcal{V}^-(\omega) \ \Rightarrow \ (1)$  has at least one positive solution

$$u^{\prime\prime}=p(t)u+h(t)|u|^{\lambda}\,\mathrm{sgn}\,u;\quad u(0)=u(\omega),\,\,u^{\prime}(0)=u^{\prime}(\omega)\,\,\Big|$$

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 $\triangleright \ p \in \mathcal{V}^{-}(\omega) \ \Rightarrow \ (1) \text{ has at least one positive solution}$ •  $p \in \mathcal{V}^{-}(\omega) \ \Rightarrow \ \exists r > 0 \text{ such that } p - \frac{h}{r^{1-\lambda}} \in \operatorname{Int} \mathcal{V}^{+}(\omega)$ 

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- $p \in \mathcal{V}^{-}(\omega) \Rightarrow \exists$  an arbitrarily large positive upper function  $\beta$  of problem (1)
- $(\alpha, \beta)$  is a couple of well-ordered lower and upper functions of (1)

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m sgn} \, u; \quad u(0) = u(\omega), \, \, u^{\prime}(0) = u^{\prime}(\omega) \, igg|$$

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- $(\alpha, \beta)$  is a couple of well-ordered lower and upper functions of (1)
- (1) has a solution u such that

$$0 < lpha(t) \le u(t) \le eta(t)$$
 for  $t \in [0, \omega]$  (5)

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 $arpropto p \in \mathcal{V}^-(\omega) \;\; \Rightarrow \;\;$  (1) has at most one positive solution

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

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 $arphi \ p \in \mathcal{V}^-(\omega) \ \Rightarrow \ (1)$  has at most one positive solution

• assume the contrary: (1) has two distinct positive solutions

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- assume the contrary: (1) has two distinct positive solutions
- $p \in \mathcal{V}^{-}(\omega) \Rightarrow (1)$  has solutions u, v such that

 $0 < u(t) \leq v(t) \quad ext{for } t \in [0, \omega], \qquad u 
ot \equiv v$ 

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

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(1) Assume that

$$u(t) < v(t)$$
 for  $t \in [0, \omega]$ .

• u, v are positive periodic solutions, respectively, to equations

$$egin{aligned} &z'' = ig( p(t) + h(t) v^{\lambda-1}(t) ig) z + h(t) ig[ u^{\lambda-1}(t) - v^{\lambda-1}(t) ig] u(t) \ &z'' = ig( p(t) + h(t) v^{\lambda-1}(t) ig) z \end{aligned}$$

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

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• the third Fredholm's theorem  $\Rightarrow$ 

$$0 = \int_0^{\omega} h(s) \left[ u^{\lambda-1}(s) - v^{\lambda-1}(s) \right] u(s) v(s) \mathrm{d}s \le Const. \int_0^{\omega} h(s) \mathrm{d}s < 0$$

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

 $arphi \; p \in \mathcal{V}^-(\omega) \;\; \Rightarrow \;\; (1)$  has at least one positive solution

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 for  $t \in [0, \omega], \qquad u 
ot\equiv v$ 

(2) Assume that

$$u(t_*)=v(t_*) ext{ for some } t_*\in [0,\omega].$$

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

 $arphi \; p \in \mathcal{V}^-(\omega) \;\; \Rightarrow \;\; (1)$  has at least one positive solution

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$$0 < u(t) \leq v(t)$$
 for  $t \in [0, \omega], \qquad u 
ot\equiv v$ 

(2) Assume that

$$u(t_*)=v(t_*)$$
 for some  $t_*\in [0,\omega].$ 

• w(t) := u(t) - v(t) is a solution of the problem

$$w^{\prime\prime}=p(t)w+h(t)\left[u^{\lambda}(t)-v^{\lambda}(t)
ight]$$

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

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ot\equiv v$ 

(2) Assume that

$$u(t_*)=v(t_*) \quad ext{for some } t_*\in [0,\omega].$$

• w(t) := u(t) - v(t) is a solution of the problem

$$w^{\prime\prime}=p(t)w+h(t)ig[u^{\lambda}(t)-v^{\lambda}(t)ig]$$

• if  $h(\cdot)[u^{\lambda}(\cdot) - v^{\lambda}(\cdot)] \equiv 0$ , then  $p \in \mathcal{V}^{-}(\omega) \Rightarrow w \equiv 0$  - contradiction

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \,\, u'(0) = u'(\omega)$$

 $arphi \; p \in \mathcal{V}^-(\omega) \;\; \Rightarrow \;\; (1)$  has at least one positive solution

 $arphi \ p \in \mathcal{V}^-(\omega) \ \Rightarrow \ (1)$  has at most one positive solution

- assume the contrary: (1) has two distinct positive solutions
- $p \in \mathcal{V}^-(\omega) \;\;\Rightarrow\;\; (1)$  has solutions u, v such that

$$0 < u(t) \leq v(t)$$
 for  $t \in [0, \omega], \qquad u 
ot\equiv v$ 

(2) Assume that

$$u(t_*)=v(t_*) \quad ext{for some } t_*\in [0,\omega].$$

• w(t) := u(t) - v(t) is a solution of the problem

$$w^{\prime\prime}=p(t)w+h(t)\left[u^{\lambda}(t)-v^{\lambda}(t)
ight]$$

• if  $h(\cdot) \left[ u^{\lambda}(\cdot) - v^{\lambda}(\cdot) \right] \equiv 0$ , then  $p \in \mathcal{V}^{-}(\omega) \implies w \equiv 0$  - contradiction

• if  $h(\cdot) \left[ u^{\lambda}(\cdot) - v^{\lambda}(\cdot) \right] \not\equiv 0$ , then  $p \in \mathcal{V}^{-}(\omega) \Rightarrow w(t) < 0$  on  $t[0, \omega]$  – contradiction

#### うしん 同一人間を入所する (四) (コ)

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 $arphi \ p \in \mathcal{V}^-(\omega) \ \Rightarrow \ (1)$  has at least one positive solution

 $arpropto p \in \mathcal{V}^-(\omega) \;\; \Rightarrow \;\;$  (1) has at most one positive solution

arproperto (1) has a positive solution  $\Rightarrow p \in \mathcal{V}^-(\omega)$ 

