p-Trigonometric and p-Hyperbolic Functions in Real and Complex Domain

Lukáš Kotrla

Supervisor: Petr Girg

Department of Mathematics and NTIS, University of West Bohemia

Czech-Georgian Workshop on Boundary Value Problems 2016

Brno, 10^{th} February 2016

For $p > 1$, we define $\sin_p$ as the solution of initial value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' - \lambda|u|^{p-2}u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = p - 1$.

Function sin (i.e. $p = 2$) is the solution of initial value problem

$$\begin{cases} -u'' - \lambda u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = 1$.

For $p > 1$, we define $\sin_p$ as the solution of initial value problem

$$\left\{ \begin{array}{rcl} -\left(|u'|^{p-2}u'\right)' - \lambda|u|^{p-2}u & = & 0\,, \\ u(0)=0, \quad u'(0) & = & 1\,, \end{array} \right.$$

for $\lambda = p - 1$.
Analogously we define $\sinh_p$ as the solution of initial value problem

$$\left\{ \begin{array}{rcl} -\left(|u'|^{p-2}u'\right)' + \lambda|u|^{p-2}u & = & 0\,, \\ u(0)=0, \quad u'(0) & = & 1\,, \end{array} \right.$$

for $\lambda = p - 1$.

Function sin (i.e. $p = 2$) is the solution of initial value problem

$$\left\{ \begin{array}{rcl} -u'' - \lambda u & = & 0\,, \\ u(0)=0, \quad u'(0) & = & 1\,, \end{array} \right.$$

for $\lambda = 1$.
Function sinh (i.e. $p = 2$) is the solution of initial value problem

$$\left\{ \begin{array}{rcl} -u'' + \lambda u & = & 0\,, \\ u(0)=0, \quad u'(0) & = & 1\,, \end{array} \right.$$

for $\lambda = 1$.

For $p > 1$, we define $\sin_p$ as the solution of initial value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' - \lambda|u|^{p-2}u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = p - 1$.
Analogously we define $\sinh_p$ as the solution of initial value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' + \lambda|u|^{p-2}u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = p - 1$.
We also define

$$\pi_p \stackrel{\text{def}}{=} 2\sup\{s > 0 \colon \forall x \in (0, s) : \sin_p(x) > 0 \wedge \sin_p'(x) > 0\}$$

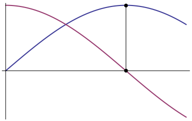
Function sin (i.e. $p = 2$) is the solution of initial value problem

$$\begin{cases} -u'' - \lambda u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = 1$.
Function sinh (i.e. $p = 2$) is the solution of initial value problem

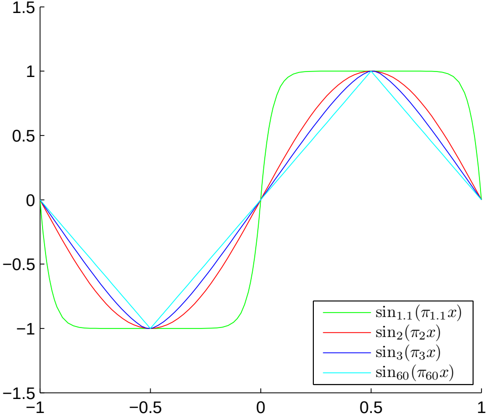
$$\begin{cases} -u'' + \lambda u &= 0\,, \\ u(0) = 0, \quad u'(0) &= 1\,, \end{cases}$$

for $\lambda = 1$.



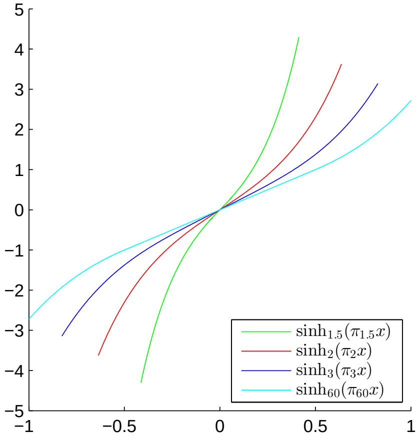
Function $\sin_p$:

- continuity
- $2\pi_p$-periodicity
- oddness
- reflection $\sin_p(x) = \sin_p(\frac{\pi_p}{2} - x)$
- $\sin_p(0) = 0$
- $\sin_p\left(\frac{\pi_p}{2}\right) = 1$
- increasing on $\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)$
- decreasing on $\left(\frac{\pi_p}{2}, \pi_p\right)$ and $\left(-\pi_p, -\frac{\pi_p}{2}\right)$
- p-trigonometric identity $|\sin_p(x)|^p + |\sin_p'(x)|^p = 1$ on $\mathbb{R}$



Function $\sinh_p$:

- continuity
- oddness
- $\sinh_p(0) = 0$
- increasing on $\mathbb{R}$
- p-hyperbolic identity $|\sinh_p'(x)|^p - |\sinh_p(x)|^p = 1$ on $\mathbb{R}$



Question

How to effectively compute $\sin_p$*?*

Question

How to effectively compute $\sin_p$*?*

Idea:
Using Maclaurin series.

Question

How to effectively compute $\sin_p$*?*

Idea:
Using Maclaurin series.

Generalized Maclaurin series: $\sum_{i=0}^{+\infty} a_i \cdot x \cdot |x|^{i \cdot r}, \; r \geq 1.$
For $\sin_p$, it converge on "small"neighborhood of 0 [Paredes, Uchiyama] (2003)

Question

How to effectively compute $\sin_p$*?*

Idea:
Using Maclaurin series.

Generalized Maclaurin series: $\sum_{i=0}^{+\infty} a_i \cdot x \cdot |x|^{i \cdot r},\ \ r \geq 1.$
For $\sin_p$, it converge on "small" neighborhood of 0 [Paredes, Uchiyama] (2003)

Question

Find the convergence rate of Maclaurin series for $\sin_p$*.*

Differentiability of $\sin_p$:

- $p = 2$: $\sin_2 \in C^\infty(\mathbb{R})$
- $p \neq 2$: $\sin_p \notin C^\infty(\mathbb{R})$
- $p > 1$: $\sin_p \in C^\infty\left(0, \frac{\pi_p}{2}\right)$
- $p = 2(m+1),\ m \in \mathbb{N}$: $\sin_{2(m+1)} \in C^\infty\left(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\right)$
- $p = \mathbb{R} \setminus \{2(m+1):\ m \in \mathbb{N}\}$: $\sin_p \in C^{\lceil p \rceil}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)$

Differentiability of $\sin_p$:

- $p = 2$: $\sin_2 \in C^\infty(\mathbb{R})$
- $p \neq 2$: $\sin_p \notin C^\infty(\mathbb{R})$
- $p > 1$: $\sin_p \in C^\infty\left(0, \frac{\pi_p}{2}\right)$
- $p = 2(m+1)$, $m \in \mathbb{N}$: $\sin_{2(m+1)} \in C^\infty\left(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\right)$
- $p = \mathbb{R} \setminus \{2(m+1) :\ m \in \mathbb{N}\}$: $\sin_p \in C^{\lceil p \rceil}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)$

Maclaurin series for $p = 2(m+1)\,, m \in \mathbb{N}$:

$$\sin_p(x) = \sum_{n=0}^{+\infty} \frac{\sin_p^{(np+1)}(0)}{(np+1)!} x^{np+1}\,, \qquad x \in \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)\,.$$

Differentiability of $\sin_p$:

- $p=2$: $\sin_2 \in C^\infty(\mathbb{R})$
- $p \neq 2$: $\sin_p \notin C^\infty(\mathbb{R})$
- $p>1$: $\sin_p \in C^\infty\left(0, \frac{\pi_p}{2}\right)$
- $p=2(m+1),\ m\in\mathbb{N}$: $\sin_{2(m+1)} \in C^\infty\left(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\right)$
- $p=\mathbb{R}\setminus\{2(m+1):\ m\in\mathbb{N}\}$: $\sin_p \in C^{\lceil p\rceil}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right)$

Maclaurin series for $p=2(m+1)\,, m\in\mathbb{N}$:

$$\sin_p(x)=\sum_{n=0}^{+\infty}\frac{\sin_p^{(np+1)}(0)}{(np+1)!}x^{np+1}\,,\qquad x\in\left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right).$$

Generalized Maclaurin series for $p=2m+1\,, m\in\mathbb{N}$:

$$\sin_p(x)=\sum_{n=0}^{+\infty}\frac{\lim_{x\to 0+}\sin_p^{np+1}(x)}{(np+1)!}x|x|^{np}\,,\qquad x\in\left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right).$$

The idea of proving the order of differentiability

- induction
- from $(uv)' = u'v + uv'$ and $\sin_p''(x) = -\sin_p^{p-1}(x)\cos_p^{2-p}(x)$ follows $\forall n \in \mathbb{N} : \sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x)\cos_p^{1-q_{k,n}}(x)$ on $\left(0, \frac{\pi_p}{2}\right)$
- we use oddness or evenness to extension on $\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right) \setminus \{0\}$
- we show that $\forall p \in \mathbb{N}, p > 1\,, \forall k, n \in \mathbb{N} : q_{k,n} \geq 0$
- from one-side limits at 0 follows continuity or discontinuity of n-th derivative of function $\sin_p$

Open problem (Convergence for $p > 1$ not integer)

Consider $p > 1$, $p \notin \mathbb{N}$. Prove (or find a counterexample) that the generalized Maclaurin series corresponding to $\sin_p$ converges on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ towards the values of $\sin_p$.

Open problem (Endpoints of the interval)

Consider $p > 1$, $p \in \mathbb{N}$. Prove (or find a counterexample) that the generalized Maclaurin series of $\sin_p$ converge at $-\frac{\pi_p}{2}$ and/or $\frac{\pi_p}{2}$.

Definitions via differential equations in complex domain:

Function $sin(z)$:

$$\begin{cases} u'' + u = 0, & z \in \mathbb{C}, \\ u(0) = 0, \\ u'(0) = 1. \end{cases}$$

Function $\sinh(z)$:

$$\begin{cases} u'' - u = 0, & z \in \mathbb{C}, \\ u(0) = 0, \\ u'(0) = 1. \end{cases}$$

Well known identity:

$$\sin(z) = -\mathrm{i}\sinh(\mathrm{i}z).$$

Definitions via differential equations in complex domain:

Function $sin(z)$:

$$\begin{cases} u'' + u = 0, & z \in \mathbb{C}, \\ u(0) = 0, & \\ u'(0) = 1. & \end{cases}$$

Function $\sinh(z)$:

$$\begin{cases} u'' - u = 0, & z \in \mathbb{C}, \\ u(0) = 0, & \\ u'(0) = 1. & \end{cases}$$

Well known identity:

$$\sin(z) = -\mathrm{i}\sinh(\mathrm{i}z).$$

Question

Does there exist an analogous identity in the case $p = 2(m+1),\ m \in \mathbb{N}$?

Remark (Complex argument for p even)

For $p = 2(m+1)$, the function $\sin_p$ can be extend to the absolutely convergent Maclaurin series. Due to this fact we can naturally extend the range of definition of $\sin_p$ to the complex open disc

$$B_p = \left\{ z \in \mathbb{C} : |z| < \frac{\pi_p}{2} \right\}.$$

Definitions via differential equations in complex domain:

- Function $sin_p(z)$:

$$\begin{cases} (u')^{p-2}u'' + u^{p-1} &= 0, \quad z \in B_p\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,. \end{cases}$$

- Function $\sinh_p(z)$:

$$\begin{cases} (u')^{p-2}u'' - u^{p-1} &= 0\,, \quad z \in D_p\,, \\ u(0) &= 0 \\ u'(0) &= 1\,. \end{cases}$$

- Let consider initial value problem for $p > 2$ even

$$\begin{cases} -(u')^{p-2}\,u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,, \end{cases}$$

in sense of differential equation in complex domain.

- It is equivalent to the first order system

$$\begin{cases} u' &= v\,, \\ v' &= -u^{p-1}/v^{p-2}\,, \\ u(0) &= 0\,, \\ v(0) &= 1\,. \end{cases} \tag{1}$$

- Main idea - show that the first order system (1) have local solution given by Maclaurin series and that this series must be the same one as the Maclaurin series for $\sin_p(x)$ is.

Theorem

Let $p = 4l + 2,\ l \in \mathbb{N}$. *Then*

$$\sin_p(z) = -\mathrm{i}\sinh_p(\mathrm{i}z)$$

for all $z \in B_p$. *Moreover,*

$$\sinh_p(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\sin^{(kp+1)}(0)}{(kp+1)!} z^{kp+1}\,.$$

Theorem

Let $p = 4l, \; l \in \mathbb{N}$. *Then*

$$\sin_p(z) = -\mathrm{i}\sin_p(\mathrm{i}z)$$

for all $z \in B_p$.

Theorem

Let $p = 4l,\ l \in \mathbb{N}$. *Then*

$$\sin_p(z) = -\mathrm{i}\sin_p(\mathrm{i}z)$$

for all $z \in B_p$.

Theorem

Let $p = 4l,\ l \in \mathbb{N}$. *Then*

$$\sinh_p(z) = -\mathrm{i}\sinh_p(\mathrm{i}z)$$

for all $z \in D_p$.

Definition of $\sin_p(z)$ for $p > 1$ odd in complex domain by Lindqvist:

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1\,. \tag{2}$$

Definition of $\sin_p(z)$ for $p > 1$ odd in complex domain by Lindqvist:

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1\,. \qquad (2)$$

Question

Is the solution of (2) restricted to $\mathbb{R}$ equal to function $\sin_p(x)$?

Definition of $\sin_p(z)$ for $p > 1$ odd in complex domain by Lindqvist:

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1\,. \tag{2}$$

Question

Is the solution of (2) *restricted to* $\mathbb{R}$ *equal to function* $\sin_p(x)$*?*

The same approach as in the case p even reveals that solution of (2) is equal to generalized Maclaurin series of function $\sin_p(x)$ only for $x \in (0, \pi_p/2)$.

Definition of $\sin_p(z)$ for $p > 1$ odd in complex domain by Lindqvist:

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0\,, \quad w(0) = 0\,, \quad w'(0) = 1\,. \qquad (2)$$

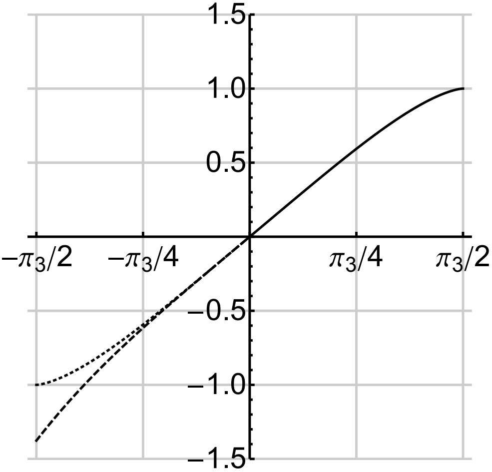
Question

Is the solution of (2) *restricted to* $\mathbb{R}$ *equal to function* $\sin_p(x)$*?*

The same approach as in the case p even reveals that solution of (2) is equal to generalized Maclaurin series of function $\sin_p(x)$ only for $x \in (0, \pi_p/2)$.

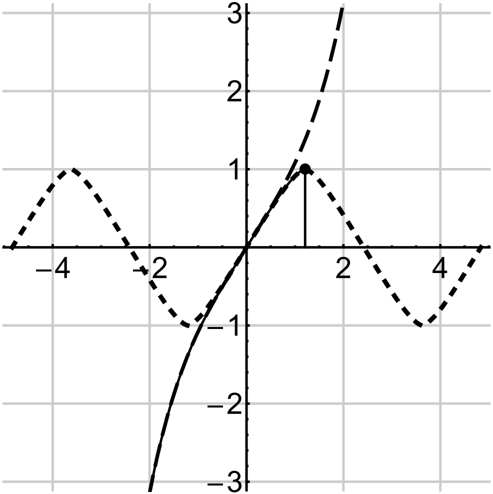
Theorem

Let $p > 1$ *is odd. Then the unique solution* $u(z)$ *of the complex initial value problem* (2) *differs from the real function* $\sin_p(x)$ *for any* $z = x \in (-\pi_p/2, 0)$.



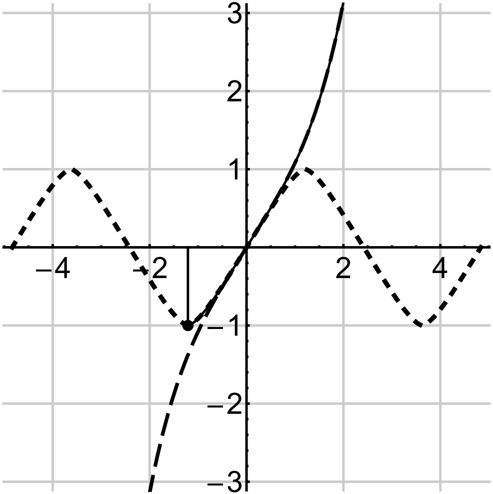
Comparison:

- restriction of complex $\sin_3(z)$ to real domain(long dashed curve)
- real function $\sin_3(x)$ (short dashed curve)



Comparison:

- restriction of complex $\sin_3(z)$ to real domain(solid curve)
- real function $\sin_3(x)$ (short dashed curve)
- real function $\sinh_3(x)$ (long dashed curve)



Comparison:

- restriction of complex $\sinh_3(z)$ to real domain(solid curve)
- real function $\sin_3(x)$ (short dashed curve)
- real function $\sinh_3(x)$ (long dashed curve).

References:

- Elbert, Á.: A half-linear second order differential equation. Qualitative theory of differential equations, Vol. I,II (Szeged, 1979), pp. 153 *Colloq. Math. Soc. János Bolyai*, **30**, North-Holland, Amsterdam-New York, 1981.
- Lindqvist, P.: NOTE ON A NONLINEAR EIGENVALUE PROBLEM, *Rocky Mountains Journal of Mathematics*, **23**, no. 1 (1993), pp. 281–288.
- Paredes, L.I.; Uchiyama, K.: Analytic Singularities of Solutions to Certain Nonlinear Ordinary Differential Equations Associated with p-Laplacian. *Tokyo J. Math.* **26**, no. 1 (2003), pp. 229–240.
- Peetre, J.: The differential equation $y'^p - y^p = \pm(p > 0)$. *Ricerche Mat.* **23** (1994), pp. 91–128.