Oscillatory Solutions of Higher Order Sublinear Differential Equations

Ivan Kiguradze[1], **Tariel Kiguradze**[2]

[1] *A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia*
E-mail: `kig@rmi.ge`

[2] *Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida, USA*
E-mail: `tkigurad@fit.edu`

We have investigated the problem on existence of oscillatory solutions of the differential equation

$$u^{(n)} = f(t,u), \tag{1}$$

where $n \geq 2$, and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function on the set $\mathbb{R}_+ \times \mathbb{R}$ satisfying either of the following two inequalities

$$f(t,x)x \leq 0 \tag{2}$$

and

$$f(t,x)x \geq 0. \tag{3}$$

A solution u of equation (1) defined on some interval $[a,+\infty[\, \subset \mathbb{R}_+$ is said to be ***proper***, if it does not identically equal to zero in any neighborhood of $+\infty$.

A proper solution $u : [a,+\infty[\, \to \mathbb{R}$ is said to be ***oscillatory***, if it changes its sign in any neighborhood of $+\infty$, and is called ***Kneser solution***, if

$$(-1)^i u^{(i)}(t)u(t) \geq 0 \ \text{ for } \ t \geq 0 \ \ (i = 1,\dots,n-1).$$

Definition 1. A continuous function$f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ belongs to the set $\mathbf{U}_0$ if equation (1) for any $t_0 \geq 0$ has only a trivial solution satisfying the initial conditions

$$u^{(i-1)}(t_0) = 0 \ \ (i = 1,\dots,n).$$

Definition 2. A continuous function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ belongs to the set $\mathbf{M}$ if there exist numbers $a > 0$ and $\lambda \in [0,1[$ such that for an arbitrarily fixed $t \in \mathbb{R}_+$ the function $x \to |f(t,x)|\operatorname{sign}(x)$ is nondecreasing in the interval $[-a,a]$, and the function $x \to |x|^{-\lambda}|f(t,x)|\operatorname{sign}(x)$ is nonincreasing in the intervals $]-\infty,a]$ and $[a,+\infty[$.

The oscillation theorems proved by us deal with the case where equation (1) is sublinear, more precisely, when

$$f \in \mathbf{M}. \tag{4}$$

Theorem 1. *Let n be even and conditions* (2) *and* (4) *hold. Then all proper solutions of equation* (1) *are oscillatory if and only if*

$$\int_0^{+\infty} t^{n-1}|f(t,x)|\,dt = +\infty, \quad \int_0^{+\infty} \big|f(t,xt^{n-1})\big|\,dt = +\infty \ \ \text{for} \ \ x \neq 0. \tag{5}$$

Theorem 2. *Let n be odd and conditions* (2) *and* (4) *hold. Then every proper solution of equation* (1) *is either oscillatory or Kneser solution if and only if*

$$\int_0^{+\infty} \big|f(t,xt^{n-1})\big|\,dt = +\infty \ \ \text{for} \ \ x \neq 0. \tag{6}$$

For equation (1), consider the initial conditions

$$u^{(i-1)}(0) = c_i \ \ (i = 1, \dots, n), \tag{7}$$

where $c_1, \dots, c_n$ are arbitrary real numbers.

Theorems 1 and 2 imply

Corollary 1. *Let n be even (odd) and let along with (2) and (5) (along with (2) and (6)) the conditions*

$$f \in \mathbf{M} \cap \mathbf{U}_0 \tag{8}$$

and

$$\sum_{i=1}^{n} |c_i| > 0 \quad \Big(c_1 = 0, \quad \sum_{i=2}^{n} |c_i| > 0 \Big) \tag{9}$$

hold. Then every solution of problem (1), (7) is oscillatory.

Let m be an integer part of the number $\frac{n}{2}$. Consider equation (1) with the boundary conditions

$$u^{(i-1)}(0) = c_i \ \ (i = 1, \dots, m), \quad \liminf_{t \to +\infty} |u^{(m)}(t)| = 0. \tag{10}$$

Theorem 3. *Let $n = 2m$, m be even, and let along with (2) and (8) the conditions*

$$\int_0^{+\infty} t^{n-1} |f(t,x)| \, dt = +\infty, \qquad \int_0^{+\infty} t^{m} \big| f(t, xt^{m-1}) \big| \, dt = +\infty \ \ \text{for} \ \ x \neq 0$$

and

$$\sum_{i=1}^{m} |c_i| > 0 \tag{11}$$

hold. Then problem (1), (10) is solvable and its every solution is oscillatory.

Theorem 4. *Let $n = 2m + 1$, $m > 1$ be odd, and let along with (2) and (8) the conditions*

$$\int_0^{+\infty} t^{m+1} \big| f(t, xt^{m-1}) \big| \, dt = +\infty$$

and

$$c_1 = 0, \quad \sum_{i=2}^{m} |c_i| > 0 \tag{12}$$

hold. Then problem (1), (10) is solvable and its every solution is oscillatory.

Assume

$$\ell_1(x) = \ln\big(1 + |x|\big) \operatorname{sign}(x), \quad \ell_{i+1}(x) = \ln\big(1 + |\ell_i(x)|\big) \operatorname{sign}(x) \ \ (i = 1, 2, \dots),$$

and consider the differential equation

$$u^{(n)} = p(t)\big(1 + |u|\big)^{\mu} \ell_k(u), \tag{13}$$

where k is some natural number, $\mu < 1$, and $p : \mathbb{R}_+ \to]-\infty, 0]$ is a continuous function.

Theorems 1–4 yield

Corollary 2. *Let n be even $($odd$)$. Then every proper solution of equation* (13) *is oscillatory $($either oscillatory, or Kneser solution$)$ if and only if*

$$\int_0^{+\infty} (1+t)^{\mu(n-1)} \ell_k(t) p(t)\, dt = -\infty. \tag{14}$$

Moreover, if conditions (9) *and* (14) *hold, then the solution of problem* (13), (7) *is oscillatory.*

Corollary 3. *Let $n = 2m$, m be even, and let along the equality*

$$\int_0^{+\infty} (1+t)^{m+\mu(m-1)} \ell_k(t) p(t)\, dt = -\infty$$

condition (11) *hold. Then problem* (13), (10) *is solvable and its every solution is oscillatory.*

Corollary 4. *Let $n = 2m+1$, $m > 1$ be odd, and let along the equality*

$$\int_0^{+\infty} (1+t)^{1+m+\mu(m-1)} \ell_k(t) p(t)\, dt = -\infty.$$

condition (12) *holds. Then problem* (13), (10) *is solvable and its every solution is oscillatory.*

The results of the type of Theorems 1–4 have been established also in the case where the function f satisfies inequality (3).