Positive Solutions of Nonlinear Boundary Value Problems on an Infinite Interval for Two-Dimensional Singular Differential Systems

Nino Partsvania[1, 2]

[1]*A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University*

[2]*International Black Sea University, Tbilisi, Georgia*

E-mail: ninopa@rmi.ge

Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[\,$, and $\mathbb{R}_{0+} =]0, +\infty[\,$. On a positive semi-axis $\mathbb{R}_{0+}$, we consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \quad (i = 1, 2) \tag{1}$$

with the boundary condition

$$\int\limits_0^a \varphi(u_1(s))\,d\sigma(s) = c, \tag{2}$$

where c is a positive constant, $f_i : \mathbb{R}_{0+} \times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ are continuous functions, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function, and $\sigma : [0, a] \to \mathbb{R}_+$ is a nondecreasing function such that

$$\lim_{x \to +\infty} \varphi(x) = +\infty, \quad \sigma(a) - \sigma(0) = 1.$$

A continuously differentiable vector function $(u_1, u_2) : \mathbb{R}_{0+} \to \mathbb{R}^2_{0+}$, satisfying system (1) in $\mathbb{R}_{0+}$, is said to be **a positive solution** of that system.

If the component u_i of a positive solution (u_1, u_2) at the point 0 has the right-hand limit

$$u_i(0+) = \lim_{t>0,\, t\to 0} u_i(t),$$

then we put $u_i(0) = u_i(0+)$.

A positive solution (u_1, u_2) of system (1) is said to be **a positive solution of problem** (1)**,** (2) if there exists $u_1(0+)$ and equality (2) is satisfied.

If

$$f_1(t, x, y) \equiv -y, \quad f_2(t, x, y) \equiv -f(t, x, -y),$$

then the differential system (1) is equivalent to the differential equation

$$u'' = f(t, u, u'), \tag{3}$$

and condition (2) is equivalent to the condition

$$\int\limits_0^a \varphi(u(s))\,d\sigma(s) = c, \tag{4}$$

respectively. Consequently, problem $(1),(2)$ has a positive solution if and only if problem $(3),(4)$ has a so-called Kneser solution, i.e. a solution satisfying the inequalities

$$u(t) > 0, \quad u'(t) < 0 \ \text{ for } \ t \in \mathbb{R}_{0+}.$$

Problem $(1),(2)$, as problem $(3),(4)$, is said to be the nonlinear Kneser problem.

We investigate problem $(1),(2)$ in the case where system (1) is singular in the phase variables; more precisely, in the case where

$$\lim_{x\to 0} f_i(t,x,y) = +\infty \ \text{ for } \ t > 0, \ \ y > 0 \ \ (i = 1,2),$$

and

$$\lim_{y\to 0} f_2(t,x,y) = +\infty \ \text{ for } \ t > 0, \ \ x > 0.$$

We have obtained unimprovable in a certain sense sufficient conditions for the existence of at least one positive solution of that problem.

Acknowledgement

This work is supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).