Positive Solutions of Nonlocal Boundary Value Problems for Singular in Phase Variables Two-Dimensional Differential Systems

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Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_{0+} =]0, +\infty[$, $C([0,a];\mathbb{R})$ be the Banach space of continuous functions $u : [0,a] \to \mathbb{R}$ with the norm

$$\|u\| = \max\left\{\|u(t)\| : \quad a \le t \le b\right\},$$

$C([0,a];\mathbb{R}_+)$ be the set of all non-negative functions from $C([0,a];\mathbb{R})$.

Consider the two-dimensional differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \ \ (i = 1, 2) \tag{1}$$

with the nonlinear boundary conditions

$$\varphi(u_1) = 0, \quad u_2(a) = \psi(u_1(a)), \tag{2}$$

where $f_i :]0,a[\times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, while $\varphi : C([0,a];\mathbb{R}_+) \to \mathbb{R}_+$ is a continuous functional.

A continuous vector function $(u_1, u_2) : [0,a] \to \mathbb{R}^2_+$ is said to be **a positive solution of the differential system (1)** if it is continuously differentiable on an open interval $]0,a[$ and in this interval along with the inequalities

$$u_i(t) > 0 \ \ (i = 1, 2) \tag{3}$$

satisfies the system (1).

A positive solution of the system (1) satisfying the conditions (2) is said to be **a positive solution of the problem (1), (2).**

We investigate the problem (1), (2) in the case where the functions f_i $(i = 1, 2)$ on the set $]0, a[\, \times \mathbb{R}^2_{0+}$ admit the estimates

$$\begin{aligned} g_{10}(t) \le -x^{\lambda_1} y^{-\mu_1} f_1(t, x, y) \le g_1(t), \\ g_{20}(t) \le -x^{\lambda_2} y^{\mu_2} f_2(t, x, y) \le g_2(t), \end{aligned} \tag{4}$$

where λ_i and μ_i $(i = 1, 2)$ are non-negative constants, and $g_{i0} :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$, $g_i :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$ are continuous functions such that

$$\int_0^a g_{i0}(t)\, dt < +\infty, \quad \int_0^a g_i(t)\, dt < +\infty \ \ (i = 1, 2).$$

If $\lambda_i > 0$ for some $i \in \{1, 2\}$, then in view of (4) we have

$$\lim_{x \to 0} f_i(t, x, y) = +\infty \ \text{ for } \ 0 < t < a, \ \ y > 0.$$

And if $\mu_2 > 0$, then

$$\lim_{y \to 0} f_2(t, x, y) = +\infty \ \text{ for } \ 0 < t < a, \ \ x > 0.$$

Consequently, in both cases the system (1) has the singularity in at least one phase variable.

Boundary value problems for singular in phase variables second order nonlinear differential equations arise in different fields of natural science and are the subject of numerous studies. See, e.g., the following works and the references therein:

1. **R. P. Agarwal, D. O'Regan, and S. Staněk**. *Math. Nachr.* **281** (2008), No. 5, 612–625.

2. **J. E. Bouillet and S. M. Gomes**. *Quart. Appl. Math.* **42** (1985), No. 4, 395–402.

3. **C. De Coster and P. Habets**. *Handbook of differential equations*, 69–160, *Elsevier/North-Holland, Amsterdam*, 2004.

4. **C. De Coster and P. Habets**. *Mathematics in Science and Engineering*, 205. *Elsevier B. V., Amsterdam*, 2006.

5. **I. T. Kiguradze**. (Russian) *Diff. Uravn.* **50** (2014), No.8, 1025-1041; transl. in *Differ. Equations* **50**(2014), No.8, 1018-1034.

6. **I. T. Kiguradze, B. L. Shekhter**. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30**(1987), 105-201; transl. in: *J. Sov. Math.* **43**(1988), No.2, 2340-2417.

7. **A. G. Lomtatidze**. *Differ. Uravn.* **22** (1986), No.6, p.1092.

8. **A. G. Lomtatidze**. *Differ. Uravn.* **23** (1987), No. 10, 1685–1692.

9. **N. Partsvania and B. Půža**. *Boundary Value Problems* 2014, **2014**:147; doi: 10.1186/s13661-014-0147-x.

10. **I. Rachůnková, S. Staněk, M. Tvrdý**. *Handbook of diff. equations: ordinary diff. equations. Vol. III*, 607–722, Handb. Differ. Equ., *Elsevier/North-Holland, Amsterdam*, 2006.

11. **I. Rachůnková, S. Staněk, and M. Tvrdý**. *Contemporary Mathematics and Its Applications*, 5. *Hindawi Publishing Corporation, New York*, 2008.

12. **S. D. Taliaferro**. *Nonlinear Anal.* **3** (1979), No. 6, 897–904.

In the recent paper,

I. Kiguradze, Positive solutions of nonlocal problems for nonlinear singular differential systems. *Mem. Differential Equations Math. Phys.* **58** (2013), 135–138,

optimal conditions are obtained for the solvability of the Cauchy–Nicoletti type nonlinear problems for singular in phase variables differential systems. As for the problems of the type $(1), (2)$, they still remain unstudied in the above-mentioned singular cases. In the present paper, the attempt is made to fill this gap.

Along with the system (1) we consider the systems of differential inequalities

$$
\begin{aligned}
-u_1^{\lambda_1}(t)u_2^{-\mu_1}(t)u_1'(t) \ge g_{10}(t),\\
-u_1^{\lambda_2}(t)u_2^{\mu_2}(t)u_2'(t) \ge g_{20}(t),
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
g_{10}(t) \le -u_1^{\lambda_1}(t)u_2^{-\mu_1}(t)u_1'(t) \le g_1(t),\\
g_{20}(t) \le -u_1^{\lambda_2}(t)u_2^{\mu_2}(t)u_2'(t) \le g_2(t).
\end{aligned}
\tag{6}
$$

Let

$$
\nu_0 = \frac{\mu_1}{1+\mu_2}\,, \quad \nu = 1+\lambda_1+\lambda_2\nu_0.
$$

On the set $\Big\{(t,x,y):\ 0\le t\le a,\ x>0,\ y\ge 0\Big\}$ we introduce the functions

$$
w_{10}(t,x,y) =
$$

$$
= \left[x^{\nu}+\nu\int_t^a g_{10}(s)\left(x^{\lambda_2}y^{1+\mu_2}+(1+\mu_2)\int_s^a g_{20}(\tau)\,d\tau\right)^{\nu_0} ds\right]^{\frac{1}{\nu}},
$$

$$
w_2(t,x,y) = \left[y^{1+\mu_2}+(1+\mu_2)\int_t^a w_{10}^{-\lambda_2}(s,x,y)g_2(s)\,ds\right]^{\frac{1}{1+\mu_2}},
$$

$$
w_1(t,x,y) = \left[x^{1+\lambda_1}+(1+\lambda_1)\int_t^a w_2^{\mu_1}(s,x,y)g_1(s)\,ds\right]^{\frac{1}{1+\lambda_1}},
$$

$$
w_{20}(t,x,y) = \left[y^{1+\mu_2}+(1+\mu_2)\int_t^a w_1^{-\lambda_2}(s,x,y)g_{20}(s)\,ds\right]^{\frac{1}{1+\lambda_2}}.
$$

Note that the functions w_1, w_2, and w_{20} are defined on the set

$$
\Big\{(t,0,y):\ 0\le t\le a,\ y\ge 0\Big\}
$$

only in the case, where

$$
\int_0^a w_{10}^{-\lambda_2}(s,0,0)g_2(s)\,ds < +\infty. \tag{7}
$$

A continuous vector function $(u_1, u_2) : [0, a] \to \mathbb{R}_+^2$ is said to be **a positive solution of the system of differential inequalities (5) (of the system of differential inequalities (6))** if it is continuously differentiable on an open interval $]0, a[$ and in this interval along with the inequalities (3) satisfies the system (5) (the system (6)).

The following statements are valid.

Lemma 1. *If the system of differential inequalities* (5) *has a positive solution* (u_1, u_2), *then*

$$u_1(t) > w_{10}(t, x, y) \ \text{ for } \ 0 \le t \le a,$$

where

$$x = u_1(a), \quad y = u_2(a). \tag{8}$$

Lemma 2. *If the system of differential inequalities* (6) *has a positive solution* (u_1, u_2), *then*

$$w_{i0}(t, x, y) < u_i(t) < w_i(t, x, y) \ \text{ for } \ 0 \le t \le a \ \ (i = 1, 2),$$

where x *and* y *are numbers given by the equalities* (8).

On the basis of these lemmas we establish conditions guaranteeing, respectively, the existence or non-existence of at least one positive solution of problem (1), (2).

As this has already been said above, the theorems proven by us concern the case where the functions f_i $(i = 1, 2)$ admit the estimates (4). Moreover, everywhere below it is assumed that the functional φ is non-decreasing, i.e. for any $u \in C([0, a]; \mathbb{R}_+)$ and $u_0 \in C([0, a]; \mathbb{R}_+)$, it satisfies the inequality

$$\varphi(u + u_0) \geq \varphi(u).$$

For any non-negative constant x, we put $\varphi(x) = \varphi(u)$, where $u(t) \equiv x$.

Theorem 1. *Let*

$$\lim_{x \to +\infty} \varphi(x) = +\infty,$$

and let for some $\delta > 0$ *the inequality*

$$\varphi\big(w_1(\,\cdot\,, \delta, \psi(\delta))\big) \leq 0$$

hold. Then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If*

$$\varphi\big(w_{10}(\,\cdot\,, 0, 0)\big) > 0,$$

then the problem (1), (2) *has no positive solution.*

The particular cases of (2) are the nonlocal boundary conditions

$$\int_0^a \psi_0(u(s))\, d\sigma(s) = c, \quad u_2(a) = \psi(u_1(a)), \tag{9}$$

where $c \in \mathbb{R}$, $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, and $\sigma : [0, a] \to \mathbb{R}$ is a nondecreasing function such that

$$\sigma(a) - \sigma(0) = 1. \tag{10}$$

Theorems 1 and 2 imply the following corollary.

Corollary 1. *If*

$$\lim_{x \to +\infty} \psi_0(x) = +\infty$$

and for some $\delta > 0$ the inequality

$$c \geq \int_0^a \psi_0\big(w_1(s, \delta, \psi(\delta))\big)\, d\sigma(s) \tag{11}$$

holds, then the problem (1), (9) *has at least one positive solution. And if*

$$c < \int_0^a \psi_0\big(w_{10}(s, 0, 0)\big)\, d\sigma(s),$$

then the problem (1), (9) *has no positive solution.*

Note that due to the condition (10), for the inequality (11) to be fulfilled it is sufficient that

$$c \geq \psi_0\big(w_1(0, \delta, \psi(\delta))\big).$$

Corollary 2. *For an arbitrary $c > 0$, the differential system* (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = c, \quad u_2(a) = 0. \tag{12}$$

For $c = 0$, the problem (1), (12) becomes much more complicated, and to guarantee its solvability we have to impose additional restrictions of functions g_{i0} and g_i. More precisely, the following theorem is valid.

Theorem 3. *If*

$$\int_0^a w_{10}^{-\lambda_2}(s,0,0)g_2(s)\,ds < +\infty, \tag{13}$$

then the differential system (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = 0, \quad u_2(a) = 0. \tag{14}$$

The condition (13) in Theorem 3 is unimprovable in a certain sense.

Moreover, the following theorem is true.

Theorem 4. *If*

$$\sup\Big\{g_i(t)/g_{i0}(t):\ \ 0<t<a\Big\}<+\infty\ \ (i=1,2),$$

then for the existence of at least one positive solution of the problem $(1),(14)$ *it is necessary and sufficient the condition*

$$\int_0^a w_{10}^{-\lambda_2}(s,0,0)g_2(s)\,ds<+\infty \tag{13}$$

to be fulfilled.

Corollary 3. *Let*

$$\inf\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_{i0}(t):\ \ 0<t<a\Big\}>0\ \ (i=1,2)$$

and

$$\sup\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_i(t):\ \ 0<t<a\Big\}<+\infty\ \ (i=1,2).$$

Then for the existence of at least one positive solution of the problem $(1),(14)$ *it is necessary and sufficient the inequalities*

$$\alpha_i>-1,\ \ \beta_i>-1\ \ (i=1,2),\quad (\alpha_2+1)(1+\lambda_1)>(\alpha_1+1)\lambda_2$$

to be satisfied.

Theorems 3, 4 and Corollary 2 are analogs of the theorems by I. Kiguradze for two-dimensional differential systems from

I. Kiguradze, The Cauchy problem for singular in phase variables nonlinear ordinary differential equations. *Georgian Math. J.* **20** (2013), No. 4, 707–720.