Positive Solutions of Nonlocal Boundary Value Problems for Singular in Phase Variables Two-Dimensional Differential Systems

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Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_{0+} =]0, +\infty[$, and $f_i : [0, a] \times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ be measurable in the first and continuous in the last two arguments functions.

Consider the two-dimensional differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \quad (i = 1, 2) \tag{1}$$

with the nonlinear nonlocal boundary conditions

$$\varphi(u_1(a_1), \dots, u_1(a_m)) = c, \quad u_2(a) = \psi(u_1(a)), \tag{2}$$

where $c \geq 0$, $0 \leq a_k \leq a$ $(k = 1, \dots, m)$, $\varphi : [0, a] \times \mathbb{R}^m_+ \to \mathbb{R}_+$ is a continuous and nondecreasing in the last m arguments function, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

An absolutely continuous vector function $(u_1, u_2) : [0, a] \to \mathbb{R}^2_+$ is said to be **a positive solution of the differential system** (1) if it satisfies the inequalities

$$u_i(t) > 0 \text{ for } 0 < t < a \quad (i = 1, 2),$$

and almost everywhere on $]0, a[$ satisfies the system (1).

A positive solution of the system (1) satisfying the conditions (2) is said to be **a positive solution of the problem** (1)**,** (2)**.**

We investigate the problem (1), (2) in the case where the functions f_i $(i = 1, 2)$ on the set $]0, a[\, \times \mathbb{R}^2_{0+}$ admit the estimates

$$\begin{gathered} g_{10}(t) \leq -x^{\lambda_1} y^{-\mu_1} f_1(t, x, y) \leq g_1(t), \\ g_{20}(t) \leq -x^{\lambda_2} y^{\mu_2} f_2(t, x, y) \leq g_2(t), \end{gathered} \tag{3}$$

where λ_i and μ_i $(i = 1, 2)$ are non-negative constants, and $g_{i0} :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$, $g_i :]0, a[\to \mathbb{R}_{0+}$ $(i = 1, 2)$ are integrable functions.

If $\lambda_i > 0$ for some $i \in \{1, 2\}$, then in view of (3) we have

$$\lim_{x \to 0} f_i(t, x, y) = +\infty \quad \text{for} \quad 0 < t < a, \quad y > 0.$$

And if $\mu_2 > 0$, then

$$\lim_{y \to 0} f_2(t, x, y) = +\infty \quad \text{for} \quad 0 < t < a, \quad x > 0.$$

Consequently, in both cases the system (1) has the singularity in at least one phase variable.

Boundary value problems for singular in phase variables second order nonlinear differential equations arise in different fields of natural science and are the subject of numerous studies (see e.g. [1], [4]–[8] and the references therein). In the recent paper by I. Kiguradze [2], optimal conditions are obtained for the solvability of the Cauchy–Nicoletti type nonlinear problems for singular in phase variables differential systems. As for the problems of the type (1), (2), they still remain unstudied in the above-mentioned singular cases.

Let

$$\nu_0 = \frac{\mu_1}{1 + \mu_2}, \qquad \nu = 1 + \lambda_1 + \lambda_2 \nu_0.$$

On the set $\big\{(t, x, y) : \ 0 \le t \le a, \ x > 0, \ y \ge 0\big\}$ we introduce the functions

$$w_0(t, x, y) = \left[x^{\nu} + \nu \int\limits_t^a g_{10}(s) \left(x^{\lambda_2} y^{1+\mu_2} + (1+\mu_2) \int\limits_s^a g_{20}(\tau)\, d\tau\right)^{\nu_0} ds\right]^{\frac{1}{\nu}},$$

$$w(t, x, y) = \left[y^{1+\mu_2} + (1 + \mu_2) \int\limits_t^a w_0^{-\lambda_2}(s, x, y) g_2(s)\, ds\right]^{\frac{1}{1+\mu_2}},$$

$$w_1(t, x, y) = \left[x^{1+\lambda_1} + (1 + \lambda_1) \int\limits_t^a w^{\mu_1}(s, x, y) g_1(s)\, ds\right]^{\frac{1}{1+\lambda_1}}.$$

Theorem 1 *Let*

$$\lim_{x \to +\infty} \varphi(x, \dots, x) = +\infty,$$

and let for some $\delta > 0$ *the inequality*

$$c \ge \varphi\big(w_1(a_1, \delta, \psi(\delta)), \dots, w_1(a_m, \delta, \psi(\delta))\big)$$

hold. Then the problem (1), (2) *has at least one positive solution.*

Theorem 2 *If*

$$c < \varphi\big(w_0(a_1, 0, 0), \dots, w_0(a_m, 0, 0)\big),$$

then the problem (1), (2) *has no positive solution.*

Theorems 1 implies the following corollary.

Corollary 1 *For an arbitrary $c > 0$, the differential system* (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = c, \quad u_2(a) = 0. \tag{4}$$

For $c = 0$, the problem (1), (4) becomes much more complicated, and to guarantee its solvability we have to impose additional restrictions of functions g_{i0} and g_i. More precisely, the following theorem is valid.

Theorem 3 *If*

$$\int_0^a w_0^{-\lambda_2}(s,0,0)g_2(s)\,ds < +\infty, \tag{5}$$

then the differential system (1) *has at least one positive solution satisfying the conditions*

$$u_1(a) = 0, \quad u_2(a) = 0. \tag{6}$$

The condition (6) in Theorem 3 is unimprovable in a certain sense. Moreover, the following theorem is true.

Theorem 4 *If*

$$\sup\Big\{g_i(t)/g_{i0}(t): \quad 0 < t < a\Big\} < +\infty \ \ (i = 1,2),$$

then for the existence of at least one positive solution of the problem (1), (6) *it is necessary and sufficient the condition* (5) *to be fulfilled.*

Corollary 2 *Let*

$$\inf\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_{i0}(t): \quad 0 < t < a\Big\} > 0 \ \ (i = 1,2)$$

and

$$\sup\Big\{t^{-\alpha_i}(a-t)^{-\beta_i}g_i(t): \quad 0 < t < a\Big\} < +\infty \ \ (i = 1,2).$$

Then for the existence of at least one positive solution of the problem (1), (6) *it is necessary and sufficient the inequalities*

$$\alpha_i > -1, \ \ \beta_i > -1 \ \ (i = 1,2), \quad (\alpha_2 + 1)(1 + \lambda_1) > (\alpha_1 + 1)\lambda_2$$

to be satisfied.

Theorems 3, 4 and Corollary 2 are analogs of the theorems by I. Kiguradze [3] for two-dimensional differential systems.

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