On Asymptotic Classification of Solutions to The Singular Third- and Fourth-Order Emden–Fowler Equations

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An asymptotic classification of solutions to equations

$$y^{(n)} + p(x)\,|y|^k \operatorname{sgn} y = 0,\ n > 2,\ k \in \mathbb{R},\ 0 < k < 1, \tag{1}$$

with continuous function $p(x) \neq 0$ for $n = 3$ and $p(x) = p_0 \neq 0$ for $n = 4$ is given.

A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1] and [4]. Results about the existence of solutions with special asymptotic behavior are contained in [2], [3], [5]–[8]. Results about asymptotic classification of solutions to (1) with $n = 3$, $k > 0$, $k \neq 1$ and $n = 4$, $k > 1$ are given in [4] and [9]. In [10] part of results of this article are formulated.

Put

$$\alpha = \frac{n}{k-1}.$$

On Asymptotic Classification of Solutions to Emden–Fowler Singular Equations of the Third Order

Consider the differential equation

$$y''' = p(x)|y|^k \operatorname{sgn} y, \quad 0 < k < 1, \tag{2}$$

with a globally defined positive continuous function $p(x)$ having positive limits p_* and p^* as $x \to \pm\infty$. Put $\beta = \frac{3}{1-k} > 0$.

Theorem 1 *(See Fig. 1) Any maximally extended solution to equation (2) is either*

(i) the trivial solution $y(x) \equiv 0$ on $(-\infty, +\infty)$, or

(ii$_\pm$) a solution equal to zero on a semi-axis $(-\infty,\ x^*]$ and constant-sign with asymptotically power behavior on $(x^*,\ +\infty)$, namely

$$y(x) = \pm C(p(x^*))\ (x - x^*)^\beta\ (1 + o(1)) \qquad \text{as } x \to x^* + 0, \tag{3}$$

$$y(x) = \pm C(p^*)\ x^\beta\ (1 + o(1)) \qquad \text{as } x \to +\infty, \tag{4}$$

where $C(p) = \left(\dfrac{(1-k)^3\,p}{3(k+2)(2k+1)}\right)^{\frac{1}{1-k}}$, *or*

(iii) a solution equal to zero on a semi-axis $[x_*,\ +\infty)$ and oscillating on $(-\infty,\ x_*)$ with its local extremum points $(x_j)_{j\in\mathbb{Z}}$ satisfying

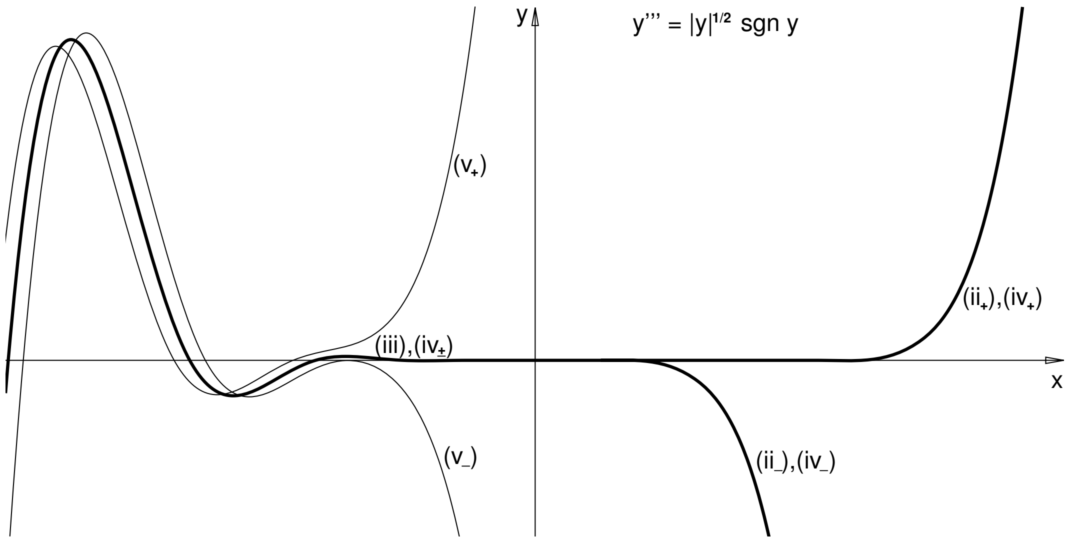
$$x_j \to -\infty, \qquad |y(x_j)| = |x_j|^{\beta+o(1)} \qquad \text{as } j \to -\infty, \tag{5}$$

$$x_j \to x_* - 0, \qquad |y(x_j)| = |x_* - x_j|^{\beta+o(1)} \qquad \text{as } j \to +\infty, \tag{6}$$

or

(iv$_\pm$) a solution equal to zero on a segment $[x_*,\ x^*]$ (the case $x_* = x^*$ is admitted), oscillating on $(-\infty,\ x_*)$ with (5)–(6) satisfied, and constant-sign on $(x^*,\ +\infty)$ with (3)–(4) satisfied, or

(v$_\pm$) a solution behaving as (4) at $+\infty$, oscillating as (5) at $-\infty$, and with no point x_0 satisfying $y(x_0) = y'(x_0) = y''(x_0) = 0$.



On Asymptotic Classification of Solutions to Emden–Fowler Singular Equations of the Fourth Order

The asymptotic classification of all possible solutions to the fourth-order Emden–Fowler type differential equations

$$y^{\mathrm{IV}}(x) + p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1,\ p_0 > 0 \tag{7}$$

and

$$y^{\mathrm{IV}}(x) - p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1,\ p_0 > 0 \tag{8}$$

is given.

In the case of regular nonlinearity $k > 1$, only maximally extended solutions are considered because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then special behavior can occur also near internal points of the domains. This is why a notion of *maximally uniquely extended (MUE)* solutions is introduced.

Definition. A solution $u : (a,\, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MUE-solution* if the following conditions hold:

(i) the equation has no solution equal to u on some subinterval of $(a,\, b)$ and not equal to u at some point of $(a,\, b)$;

(ii) either there is no solution defined on another interval containing $(a,\, b)$ and equal to u on $(a,\, b)$ or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of $(a,\, b)$.

In this article all MUE-solutions to equation (7) and (8) are classified according to their behavior near the boundaries of their domains. All maximally extended solution can be classified through investigation of possible ways to join several MUE-solutions.

Theorem 2 *Suppose* $0 < k < 1$ *and* $p_0 > 0$. *Then all MUE-solutions to equation* (7) *are divided into the following three types according to their asymptotic behavior (see Fig. 2).*

1. Oscillatory solutions defined on semi-axes $(-\infty, b)$. *The distance between their neighboring zeros infinitely increases near* $-\infty$ *and tends to zero near* b. *The solutions and their derivatives satisfy the relations* $\lim\limits_{x\to b} y^{(j)}(x) = 0$ *and* $\varlimsup\limits_{x\to-\infty} \left|y^{(j)}(x)\right| = \infty$ *for* $j = 0, 1, 2, 3$. *At the points of local extremum the following estimates hold:*

$$C_1\,|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2\,|x-b|^{-\frac{4}{k-1}} \tag{9}$$

with positive constants C_1 *and* C_2 *depending only on* k *and* p_0.

2. Oscillatory solutions defined on semi-axes $(b, +\infty)$. The distance between their neighboring zeros tends to zero near b and infinitely increases near $+\infty$. The solutions and their derivatives satisfy the relations $\lim_{x\to b} y^{(j)}(x) = 0$ and $\overline{\lim_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum estimates (9) hold with positive constants C_1 and C_2 depending only on k and p_0.

3. Oscillatory solutions defined on $(-\infty, +\infty)$. All their derivatives $y^{(j)}$ with $j = 0, 1, 2, 3, 4$ satisfy

$$\overline{\lim_{x\to-\infty}} \left|y^{(j)}(x)\right| = \overline{\lim_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty.$$

At the points of local extremum the estimates

$$C_1 \, |x|^{-\frac{4}{k-1}} \le |y(x)| \le C_2 \, |x|^{-\frac{4}{k-1}} \tag{10}$$

hold near $-\infty$ or $+\infty$ with positive constants C_1 and C_2 depending only on k and p_0.

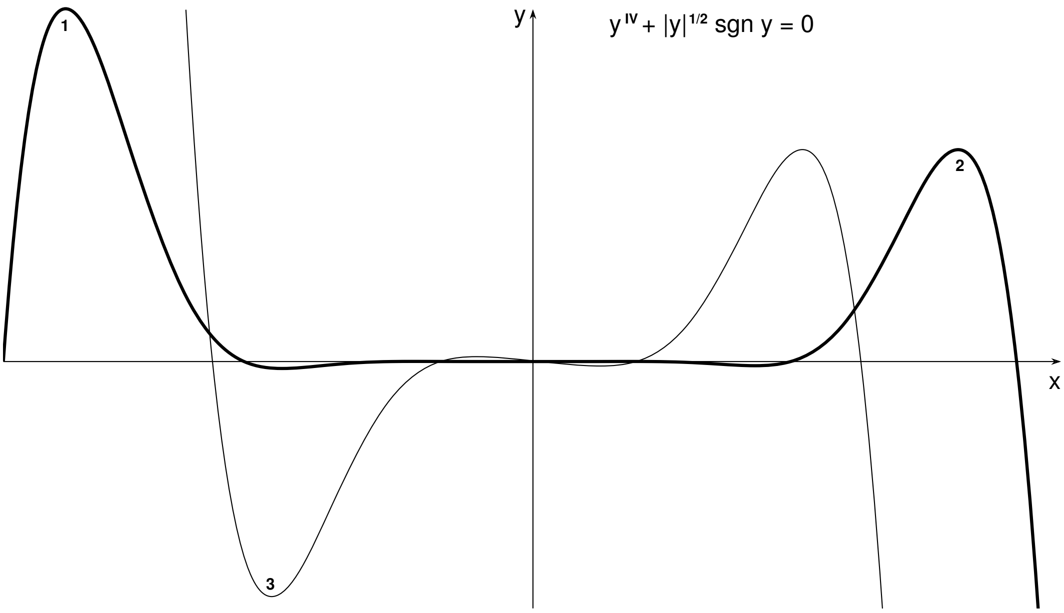


Fig. 2

Theorem 3 *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MUE-solutions to equation (8) are divided into the following thirteen types according to their asymptotic behavior (see Fig. 3).*

1–2. Defined on semi-axes $(b, +\infty)$ solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):

$$y(x) \sim \pm C_{4k} \, (x-b)^{-\frac{4}{k-1}}, \qquad x \to b+0,$$
$$y(x) \sim \pm C_{4k} \, x^{-\frac{4}{k-1}}, \qquad x \to +\infty,$$

where

$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{p_0 \, (k-1)^4} \right)^{\frac{1}{k-1}}.$$

3–4. Defined on semi-axes $(-\infty, b)$ solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs $\pm$):

$$y(x) \sim \pm C_{4k} \, |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty,$$
$$y(x) \sim \pm C_{4k} \, (b-x)^{-\frac{4}{k-1}}, \qquad x \to b-0.$$

5. Defined on the whole axis periodic oscillatory solutions. All of them can be received from one, say $z(x)$, by the relation

$$y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$$

with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–7. Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to -\infty$ and have the power asymptotic behavior near $+\infty$:

$$y(x) \sim \pm C_{4k}\, x^{-\frac{4}{k-1}}, \qquad x \to +\infty.$$

For each solution of this type a finite limit of the absolute values of its local extrema exists as $x \to -\infty$.

8–9. Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to +\infty$ and have the power asymptotic behavior near near $-\infty$:

$$y(x) \sim \pm C_{4k}\, |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty.$$

For each solution of this type a finite limit of the absolute values of its local extrema exists as $x \to +\infty$.

10–13. Defined on $(-\infty, +\infty)$ solutions having the power asymptotic behavior both near $-\infty$ and $+\infty$ (with four possible pairs of signs $\pm$):

$$y(x) \sim \pm C_{4k}(p(b))\ |x|^{-\frac{4}{k-1}}, \qquad x \to \pm\infty.$$

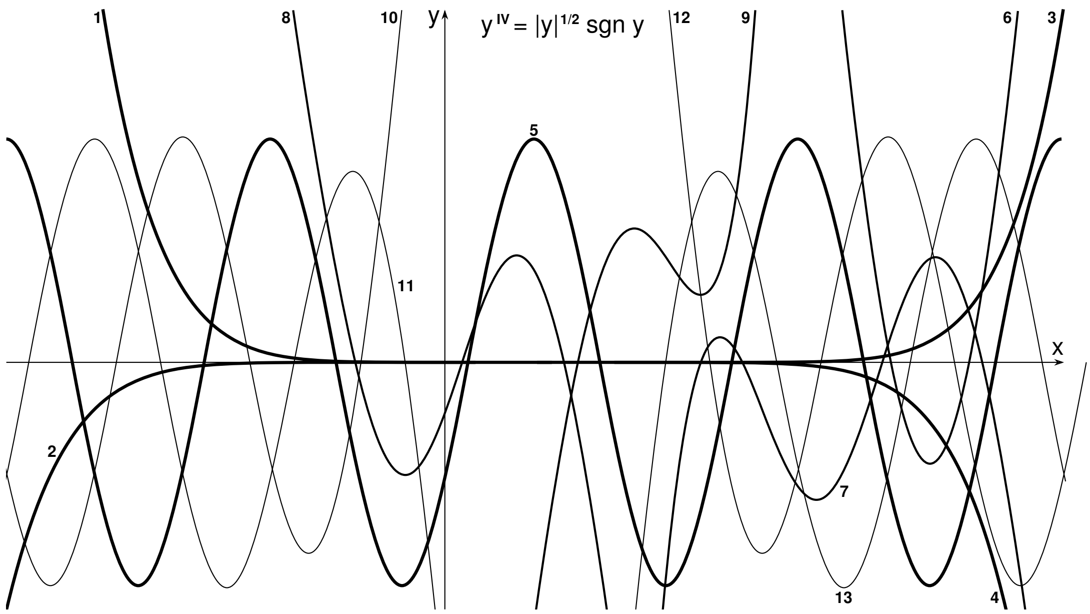


Fig. 3

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