Some oscillation criteria for second-order linear delay differential equations

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$$u''(t) + p(t)u\big(\tau(t)\big) = 0 \tag{1}$$

- $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lebesgue integrable function
- $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function such that

$$\tau(t) \leq t \quad \text{for a. e. } t \geq 0, \qquad \lim_{t\to+\infty} \operatorname{ess\,inf}\big\{\tau(s) : s \geq t\big\} = +\infty$$

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Solution:

Let $t_0 \ge 0$ and $a_0 = \operatorname{ess\,inf}\{\tau(t) : t \ge t_0\}$.

A continuous function $u\colon [a_0, +\infty[\to \mathbb{R}$ is said to be a solution to equation (1) on the interval $[t_0, +\infty[$, if it is absolutely continuous together with the first derivative on every compact interval in $[t_0, +\infty[$ and satisfies equality (1) almost everywhere in $[t_0, +\infty[$.

A particular case:

$$\boxed{u'' + p(t)u = 0} \tag{2}$$

- $p\colon \mathbb{R}_+ \to \boxed{\mathbb{R}}$ is a locally Lebesgue integrable function

Equation (2) is said to be oscillatory if every nontrivial solution $u\colon [a, +\infty[\to \mathbb{R}$ to this equation has a sequence of zeros tending to $+\infty$, and nonoscillatory otherwise.

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Put

$$c(t) := \frac{1}{t}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t > 0.$$

Theorem (Hartman-Wintner). *Let either*

$$\lim_{t\to+\infty} c(t) = +\infty$$

or

$$-\infty < \liminf_{t\to+\infty} c(t) < \limsup_{t\to+\infty} c(t)$$

Then equation (2) *is oscillatory.*

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Two cases remain uncovered in the previous theorem:

- $\liminf_{t\to+\infty} c(t) = -\infty$
- there exists a finite limit $\lim_{t\to+\infty} c(t)$

The case where there exists a finite limit $\lim_{t\to+\infty} c(t)$

$$c(t) := \frac{1}{t}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t > 0.$$

Let there exists a finite limit

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Let there exists a finite limit

$$c_0 := \lim_{t\to+\infty} c(t).$$

We put

$$q(t) := t\left(c_0 - \int_0^t p(s)\mathrm{d}s\right) \quad \text{for } t \geq 0$$

and

$$h(t) := \frac{1}{t}\int_0^t s^2 p(s)\mathrm{d}s \quad \text{for } t > 0$$

Then oscillatory properties of equation (2) can be described in terms of the numbers

$$q_* = \liminf_{t\to+\infty} q(t), \qquad q^* = \limsup_{t\to+\infty} q(t),$$

$$h_* = \liminf_{t\to+\infty} h(t), \qquad h^* = \limsup_{t\to+\infty} h(t)$$

In particular, equation (2) is oscillatory under each of the following conditions:

- E. Hille (1948) $[p(t) \geq 0]$

$$\text{either} \qquad q^* > 1 \qquad \text{or} \qquad q_* > \frac{1}{4}$$

Non-oscillation theorems, Trans. Amer. Math. Soc. **64** (1948), No. 2, 234–252.

- Z. Nehari (1957) $[p(t) \geq 0]$

$$\text{either} \qquad h^* > 1 \qquad \text{or} \qquad h_* > \frac{1}{4}$$

Oscillation criteria for second-order linear differential equations Trans. Amer. Math. Soc. **85** (1957), No. 2, 428–445.

- A. Lomtatidze, T. Chantladze, N. Kandelaki (1999)

  either

$$0 \le q_* \le \frac{1}{4}, \qquad h^* > \frac{1}{2}\left(1 + \sqrt{1 - 4q_*}\right)$$

  or

$$0 \le h_* \le \frac{1}{4}, \qquad q^* > \frac{1}{2}\left(1 + \sqrt{1 - 4h_*}\right)$$

Oscillation and nonoscillation criteria for a second order linear differential equation, Georgian Math. J. **6** (1999), No. 5, 401-414

$$\boxed{u''(t) + p(t)u\big(\tau(t)\big) = 0} \tag{1}$$

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Example. Let $t^* \in]3\pi/2, 2\pi[$ be such that

$$\frac{\sin t^*}{(t^* - 3\pi)^2} = -k, \quad \text{where} \quad k = \max\left\{-\frac{\sin t}{(t-3\pi)^2} : t \in [3\pi/2, 2\pi]\right\}.$$

Put

$$p(t) = \begin{cases} 1 & \text{for } t \in [0, t^*[\,\cup[3\pi, +\infty[\,, \\ 2k & \text{for } t \in [t^*, 3\pi], \end{cases} \qquad \tau(t) = \begin{cases} t & \text{for } t \in [0, t^*[\,\cup[3\pi, +\infty[\,, \\ \pi/2 & \text{for } t \in [t^*, 3\pi], \end{cases}$$

and

$$u(t) = \begin{cases} -\sin t & \text{for } t \in [0, t^*[\,, \\ k(t-3\pi)^2 & \text{for } t \in [t^*, 3\pi] \\ 0 & \text{for } t \in [3\pi, +\infty[\,. \end{cases}$$

$$\boxed{u''(t) + p(t)u\big(\tau(t)\big) = 0} \qquad (1)$$

Definition. A solution u to equation (1) on the interval $[t_0, +\infty[$ is said to be proper if it satisfies the relation

$$\sup\Big\{\big|u(s)\big| : s \geq t\Big\} > 0 \quad \text{for } t \geq t_0.$$

Definition. A proper solution to equation (1) is said to be oscillatory if it has a sequence of zeros tending to infinity, and non-oscillatory otherwise.

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Proposition 1

If

$$\int_0^{+\infty} sp(s)\mathrm{d}s < +\infty,$$

then equation (1) *has a proper non-oscillatory solution.*

In what follows, we assume that $\displaystyle\int_0^{+\infty} sp(s)\mathrm{d}s = +\infty.$

- Results presented below are proved by using lemmas on a priori estimates of non-oscillatory solutions. If u is a proper non-oscillatory solution to (1) satisfying

$$u(t) \neq 0 \quad \text{for large} \quad t,$$

and $\rho(t) = \frac{u'(t)}{u(t)}$ then from equation (1) we obtain

$$\rho'(t) = -p(t)\frac{u(\tau(t))}{u(t)} - \rho^2(t) \quad \text{for large} \quad t.$$

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- In the case of ODEs we have $\frac{u(\tau(t))}{u(t)} \equiv 1$ and thus ρ is a solution to the Riccati equation

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- Therefore, to extend Wintner, Hille and Nehari type oscillation criteria for equations with argument deviations as well as to prove Myshkis type criteria we need to find suitable lower and upper bounds of the quantity $\frac{u(\tau(t))}{u(t)}$.

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- Therefore, to extend Wintner, Hille and Nehari type oscillation criteria for equations with argument deviations as well as to prove Myshkis type criteria we need to find suitable lower and upper bounds of the quantity $\frac{u(\tau(t))}{u(t)}$.

- It is not difficult to show that

$$\frac{\tau(t)}{t} \leq \frac{u(\tau(t))}{u(t)} \leq 1 \quad \text{for } t \text{ large enough.}$$

Lemma on an a priori estimate

Let u *be a solution to equation* (1) *on the interval* $[t_u, +\infty[$ *such that*

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Then

$$\int_0^{+\infty} \frac{\tau(s)}{s} p(s) \mathrm{d}s < +\infty$$

and

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t s\tau(s)p(s)\mathrm{d}s \leq 1, \qquad \limsup_{t \to +\infty} t \int_t^{+\infty} \frac{\tau(s)}{s} p(s)\mathrm{d}s \leq 1.$$

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Moreover, for any $\varepsilon \in [0, 1[$, *there exists* $t_0(\varepsilon) \leq t_u$ *such that*

$$\left(\frac{T_1}{T_2}\right)^{1-\varepsilon G_*} \leq \frac{u(T_1)}{u(T_2)} \leq \left(\frac{T_1}{T_2}\right)^{\varepsilon F_*} \quad \text{for } T_2 \geq T_1 \geq t_0(\varepsilon)$$

where

$$G_* = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t s\tau(s)p(s)\mathrm{d}s, \qquad F_* = \liminf_{t \to +\infty} t \int_t^{+\infty} \frac{\tau(s)}{s} p(s)\mathrm{d}s.$$

Proposition 2

If either

$$\limsup_{t \to +\infty} t \int_t^{+\infty} \frac{\tau(s)}{s} p(s) \mathrm{d}s > 1,$$

or

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t s\tau(s)p(s) \mathrm{d}s > 1,$$

then every proper solution to equation (1) *is oscillatory.*

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$$\limsup_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\mathrm{d}s > 1,$$

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Remark. If $\tau(t) \equiv t$ then we have

$$\limsup_{t\to+\infty} t\int_t^{+\infty} p(s)\mathrm{d}s > 1 \qquad \big(i.e., \quad q^* > 1\big), \qquad \text{Hille (1948)}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t s^2 p(s)\mathrm{d}s > 1 \qquad \big(i.e., \quad h^* > 1\big) \qquad \text{Nehari (1957).}$$

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then every proper solution to equation (1) *is oscillatory.*

Therefore, we assume in the sequel that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)\mathrm{d}s \le 1, \qquad \limsup_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\mathrm{d}s \le 1,$$

and thus

$$G_* \le 1, \qquad F_* \le 1,$$

where

$$G_* = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)\mathrm{d}s, \qquad F_* = \liminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}\, p(s)\mathrm{d}s.$$

Theorem 1

Let there exist $\varepsilon \in [0,1[$ *such that*

$$\int_0^{+\infty} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)\mathrm{d}s = +\infty. \quad (3)$$

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Then every proper solution to equation (1) *is oscillatory.*

Remark. If there exists $\alpha > 0$ such that τ satisfies

$$\frac{\tau(t)}{t} \geq \alpha > 0 \quad \text{for large } t \tag{4}$$

then assumption (6) can be replaced by the more convenient assumption

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Example. In the equation with a proportional delay

$$u''(t) + p(t)u(\alpha t) = 0, \qquad 0 < \alpha \leq 1,$$

the argument deviation satisfies relation (4).

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$$\int_0^{+\infty} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)\mathrm{d}s = +\infty. \tag{3}$$

Then every proper solution to equation (1) *is oscillatory.*

Remark. Theorem 1 can be regarded as a Wintner type result because if $p(t) \geq 0$ then

$$\lim_{t\to+\infty} c(t) = +\infty$$

$$\Updownarrow$$

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s = +\infty$$

$$\Updownarrow$$

$$\int_0^{+\infty} p(s)\mathrm{d}s = +\infty \qquad \Big(\text{proved already by Fite (1918)}\Big).$$

According to Theorem 1, it is nature to suppose in what follows that

$$\int_0^{+\infty} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)\mathrm{d}s < +\infty \quad \text{for all } \varepsilon \in [0,1[\,.$$

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We put

$$Q(t;\varepsilon) := t\int_t^{+\infty} \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)\mathrm{d}s \quad \text{for } t \geq 0,$$

$$H(t;\varepsilon) := \frac{1}{t}\int_0^{t} s^2 \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} p(s)\mathrm{d}s \quad \text{for } t > 0$$

and we describe oscillatory properties of equation (1) in terms of the numbers

$$Q_*(\varepsilon) = \liminf_{t\to+\infty} Q(t;\varepsilon), \qquad Q^*(\varepsilon) = \limsup_{t\to+\infty} Q(t;\varepsilon),$$

$$H_*(\varepsilon) = \liminf_{t\to+\infty} H(t;\varepsilon), \qquad H^*(\varepsilon) = \limsup_{t\to+\infty} H(t;\varepsilon)$$

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If $p(t) \geq 0$ then

$$\lim_{t\to+\infty} c(t) = \lim_{t\to+\infty}\frac{1}{t}\int_0^t\left(\int_0^s p(\xi)\mathrm{d}\xi\right)\mathrm{d}s = c_0$$

$$\Updownarrow$$

$$\int_0^{+\infty} p(s)\mathrm{d}s = c_0.$$

and thus

$$q(t) = t\left(c_0 - \int_0^t p(s)\mathrm{d}s\right) = t\int_t^{+\infty} p(s)\mathrm{d}s \quad \text{for } t \geq 0,$$

$$h(t) = \frac{1}{t}\int_0^t s^2 p(s)\mathrm{d}s \quad \text{for } t > 0$$

Theorem 2

Let there exist a number $\varepsilon \in [0,1[$ *such that*

$$\limsup_{t\to+\infty}\Big(Q(t;\varepsilon) + H(t;\varepsilon)\Big) > 1.$$

Then every proper solution to equation (1) *is oscillatory.*

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Let there exist a number $\varepsilon \in [0,1[$ such that

$$\limsup_{t\to+\infty}\Big(Q(t;\varepsilon) + H(t;\varepsilon)\Big) > 1.$$

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As corollaries of Theorem 2 we obtain the following statements, which coincide with the classical Hille and Nehari results in the case of ODEs.

Corollary 1

Let there exist $\varepsilon \in [0,1[$ such that

$$Q^*(\varepsilon) > 1.$$

Then every proper solution to equation (1) *is oscillatory.*

Corollary 1 is Hille type result. Indeed, if $\tau(t) \equiv t$ and $p(t) \geq 0$ then

$$Q^*(\varepsilon) = \limsup_{t\to+\infty} t\int_t^{+\infty} p(s)ds > 1 \iff q^* = \limsup_{t\to+\infty} q(t) > 1$$

.

Corollary 2

Let there exist $\varepsilon \in [0,1[$ *such that*

$$H^*(\varepsilon) > 1.$$

Then every proper solution to equation (1) *is oscillatory.*

Corollary 2 is Nehari type result. Indeed, if $\tau(t) \equiv t$ then

$$H^*(\varepsilon) = h^*$$

Theorem 3

Let there exist $\varepsilon \in [0,1[$ such that

$$\liminf_{t\to+\infty}\Big(Q(t;\varepsilon)+H(t;\varepsilon)\Big) > \frac{1}{2}\,.$$

Then every proper solution to equation (1) is oscillatory.

Theorem 3 (in fact, its generalization) yields the following corollary which coincides with the classical Hille result for ODEs.

Corollary 3

Let there exist $\varepsilon \in [0,1[$ such that

$$Q_*(\varepsilon) > \frac{1}{4}\,.$$

Then every proper solution to equation (1) is oscillatory.

$$Q(t;\varepsilon) = t\int_t^{+\infty} p(s)\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} \mathrm{d}s, \qquad H(t;\varepsilon) = \frac{1}{t}\int_0^t s^2 p(s)\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} \mathrm{d}s$$

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Theorem 4

Let there exist $\varepsilon \in [0,1[$ *such that*

$$Q_*(\varepsilon) \le \frac{1}{4}\,, \qquad H^*(\varepsilon) > \frac{1}{2}\Big(1+\sqrt{1-4Q_*(\varepsilon)}\Big).$$

Then every proper solution to equation (1) *is oscillatory.*

$$Q(t;\varepsilon) = t\int_t^{+\infty} p(s)\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} ds, \qquad H(t;\varepsilon) = \frac{1}{t}\int_0^t s^2 p(s)\left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} ds$$

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Let there exist $\varepsilon \in [0, 1[$ *such that*

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Theorem 3 (in fact, its generalization) also yields

Corollary 4

Let there exist $\varepsilon \in [0, 1[$ *such that*

$$H_*(\varepsilon) > \frac{1}{4}\,.$$

Then every proper solution to equation (1) *is oscillatory.*

Theorem 5

Let there exist $\varepsilon \in [0, 1[$ *such that*

$$H_*(\varepsilon) \le \frac{1}{4}\,, \qquad Q^*(\varepsilon) > \frac{1}{2}\left(1 + \sqrt{1 - 4H_*(\varepsilon)}\right).$$

Then every proper solution to equation (1) *is oscillatory.*

Remark. All previous results we can formulate in more general way.

$$Q(t;\lambda,\varepsilon) = t^{1-\lambda} \int_t^{+\infty} s^{\lambda} p(s) \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} \mathrm{d}s \quad \text{for } t \geq 0,$$

$$H(t;\mu,\varepsilon) = \frac{1}{t^{\mu-1}} \int_0^t s^{\mu} p(s) \left(\frac{\tau(s)}{s}\right)^{1-\varepsilon G_*} \mathrm{d}s \quad \text{for } t > 0,$$

where $\lambda < 1$ and $\mu > 1$.

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Theorem 3*

Let there exist numbers $\varepsilon \in [0,1[$ $\lambda < 1$ *and* $\mu > 1$ *such that*

$$\liminf_{t \to +\infty} \Big(Q(t;\lambda,\varepsilon) + H(t;\mu,\varepsilon) \Big) > \frac{1}{4(1-\lambda)} + \frac{1}{4(\mu-1)} .$$

Then every proper solution to equation (1) *is oscillatory.*

Example.

$$u''(t) + \frac{\gamma}{t^2}u(\alpha t) = 0 \quad \text{for } t \in [1, +\infty[,$$

where $0 < \alpha \leq 1$, $\gamma \geq 0$.

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$$\gamma > \frac{1}{4} \quad \Rightarrow \quad \text{ODE is oscillatory}$$

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- DDE (1)

$$p(t) \equiv \frac{\gamma}{t^2}, \quad \tau(t) \equiv \alpha t$$

Corollary 3 Let $G_* \leq 1$ and $Q_* > \frac{1}{4}$ $\Rightarrow$ every proper solution is oscillatory.

Proposition 2 Let $\limsup_{t\to+\infty} \frac{1}{t}\int_1^{+\infty} s\tau(s)p(s)\mathrm{d}s > 1$ $\Rightarrow$ every proper solution is oscillatory.

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- if $\gamma\alpha \leq 1$ and $\gamma\alpha^{1-\gamma\alpha} > \frac{1}{4}$ $\Rightarrow$ every proper solution is oscillatory

For the sake of simplicity in formulations (not any technical reasons!) we will assume in what follows that the delay τ is continuous.

Myshkis type oscillation criteria

- The Myshkis criterion (oscillatory) $a\Delta > \frac{1}{e}$ derived for the equation

$$u'(t) + au(t-\Delta) = 0.$$

- For the first-order equation

$$u'(t) + p(t)u\big(\tau(t)\big) = 0,$$

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Theorem A (Koplatadze 1986). *Let there exist a continuous non-decreasing function* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
$$\tau(t) \le \sigma(t) \le t \quad \text{for } t \ge 0$$
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Then every proper solution to equation (1) *is oscillatory.*

Remark. If condition (4) is satisfied, then necessarily
$$\int_0^{+\infty} \tau(s)p(s)\mathrm{d}s = +\infty.$$

Theorem 6

Let there exist numbers $\varepsilon_1, \varepsilon_2 \in [0, 1[$ *and continuous functions* $\sigma, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\tau(t) \le \nu(t) \le \sigma(t) \le t \quad \text{for } t \ge 0,$$

σ *is non-decreasing and*

$$\limsup_{t\to+\infty} \left(\frac{\nu(t)}{\sigma(t)}\right)^{1-\varepsilon_2 F_*} \int_{\nu(t)}^{t} \tau(s)p(s) \left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon_1 G_*} \mathrm{d}s > 1. \tag{5}$$

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Then every proper solution to equation (1) *is oscillatory.*

Remark Condition (5) may be satisfied even if

$$\int_0^{+\infty} \tau(s)p(s)\mathrm{d}s < +\infty.$$

However, in such a case it is necessary that

$$\limsup_{t\to+\infty} \frac{\sigma(t)}{\tau(t)} = +\infty, \qquad \varepsilon_1 G_* > 0,$$

e. g.,

$$\tau(t) := \max\{t \sin t, t^{\alpha}\}, \quad \sigma(t) := t \quad \text{for large } t,$$

where $0 < \alpha < 1$.

Theorem 7

Let there exist numbers $\varepsilon_1, \varepsilon_2 \in [0,1[$ *and continuous functions* $\sigma, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* σ *is non-decreasing,*

$$\tau(t) \le \nu(t) \le \sigma(t) \le t \quad \text{for } t \ge 0, \tag{6}$$

$$Q_*(\varepsilon_1) \le \frac{1}{4}, \quad H_*(\varepsilon_1) \le \frac{1}{4} \tag{7}$$

and

$$\limsup_{t\to+\infty} \left(\frac{\nu(t)}{\sigma(t)}\right)^{1-\varepsilon_2 F_*} \int_{\nu(t)}^t \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon_1 G_*} ds > R - \alpha r, \tag{8}$$

where

$$r = \frac{1}{2}\left(1 - \sqrt{1 - 4Q_*(\varepsilon_1)}\right), \; R = \frac{1}{2}\left(1 + \sqrt{1 - 4H_*(\varepsilon_1)}\right), \; \alpha = \liminf_{t\to+\infty}\left(\frac{\nu(t)}{t}\right)^{1-\varepsilon_2 F_*}$$

Then every proper solution to equation (1) *is oscillatory.*

If we put $\nu \equiv \sigma$ and $\varepsilon_1 = \varepsilon_2$ in Theorem 7 we obtain

Corollary 5

Let there exist number $\varepsilon_1 \in [0,1[$ and a nondecreasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that conditions (6) and (7) are fulfilled and

$$\limsup_{t\to+\infty} \int_{\sigma(t)}^t \tau(s)p(s)\left(\frac{\sigma(s)}{\tau(s)}\right)^{\varepsilon_1 G_*} ds > R - \alpha r$$

Then every proper solution to equation (1) is oscillatory.

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Corollary 5

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Then every proper solution to equation (1) is oscillatory.

Remark. Observe that $0 \le \alpha \le 1$ and

$$0 \le r \le \frac{1}{2} \le R \le 1.$$

In view of latter inequalities, it is clear that $R - \alpha r \le 1$ and thus Corollary 5 improves Theorem A, under additional assumptions

$$Q_*(\varepsilon_1) \le \frac{1}{4}, \quad H_*(\varepsilon_1) \le \frac{1}{4}. \tag{7}$$

These assumptions, in fact, do not bring any restrictions because we know that

$$Q_*(\varepsilon) > \frac{1}{4} \quad \text{or} \quad H_*(\varepsilon) > \frac{1}{4} \quad \Longrightarrow \quad \text{oscillations of regular solutions}$$

(see Corollary 3 and 4)

Theorem A (Koplatadze 1986). *Let there exist a continuous non-decreasing function* $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\tau(t) \le \sigma(t) \le t \quad \text{for } t \ge 0 \qquad (6) \qquad\qquad \limsup_{t\to+\infty}\int_{\sigma(t)}^{t} \tau(s)p(s)\mathrm{d}s > 1 \qquad (**)$$

Then every proper solution to equation (1) *is oscillatory.*

Theorem B (Koplatadze 1986). *Let*

$$\liminf_{t\to+\infty}\int_{\tau(t)}^{t}\tau(s)p(s)\mathrm{d}s > \frac{1}{\mathrm{e}}\,. \tag{9}$$

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- If condition (9) holds then, as well as before, necessarily

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In order to derive oscillation criteria of the type

$$\liminf_{t \to +\infty} \cdots \int_{\tau(t)}^{t} \ldots \mathrm{d}s > \cdots$$

from our lemma on an a priori estimate we need the following technical assumption

$$\limsup_{t \to +\infty} \frac{\tau(t)}{t} > 0.$$

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Theorem 8

Let

$$\lim_{t\to+\infty} \frac{\tau(t)}{t} = 0. \tag{10}$$

Then every proper solution to equation (1) *is oscillatory provided that* $G_* > 0$, *i. e.,*

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Remark. Condition (10) is satisfied, e. g., if

$$\tau(t) \le t^a \quad \text{for large } t,$$

where $0 < a < 1$.

Theorem 9

Let there exist number $\varepsilon \in [0,1[$ *and continuous function* $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\tau(t) \le \nu(t) \le t$ *for* $t \ge 0$,

$$Q_*(\varepsilon) \le \frac{1}{4}, \quad H_*(\varepsilon) \le \frac{1}{4}, \tag{7}$$

and

$$\liminf_{t\to+\infty} \nu^{\varepsilon G_*}(t) \int_{\nu(t)}^{t} \tau^{1-\varepsilon G_*}(s) p(s) ds > R - \beta r, \tag{11}$$

where

$$r = \frac{1}{2}\left(1 - \sqrt{1 - 4Q_*(\varepsilon)}\right), \; R = \frac{1}{2}\left(1 + \sqrt{1 - 4H_*(\varepsilon)}\right), \; \beta = \liminf_{t\to+\infty} \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*}$$

Let, moreover, $\limsup_{t\to+\infty} \frac{\tau(t)}{t} > 0$. *Then every proper solution to equation* (1) *is oscillatory.*

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Let, moreover, $\limsup_{t\to+\infty} \frac{\tau(t)}{t} > 0$. *Then every proper solution to equation* (1) *is oscillatory.*

Remark. If $\varepsilon G_* > 0$ then inequality (11) can be fulfilled even when $\int_0^{+\infty} \tau(s)p(s)ds < \infty$ and thus condition $\int_0^{+\infty} \tau(s)p(s)ds = +\infty$ necessary for the validity of assumption

$$\liminf_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds > \frac{1}{e} \tag{9}$$

in Koplatadze's Theorem B, is weakened in Theorem 9.

In the last two statements we ensure that $\beta = \liminf_{t\to+\infty} \left(\frac{\nu(t)}{t}\right)^{\varepsilon G_*}$ is equal to 1.

Corollary 6 $(\nu = \tau,\ \varepsilon = 0)$

Let $Q_*(0) \le \frac{1}{4}$, $H_*(0) \le \frac{1}{4}$

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > R_0 - r_0$$

where

$$r_0 = \frac{1}{2}\left(1 - \sqrt{1 - 4Q_*(0)}\right), \quad R_0 = \frac{1}{2}\left(1 + \sqrt{1 - 4H_*(0),}\right).$$

Then every proper solution to equation (1) *is oscillatory.*

$Q_*(0) = \liminf_{t\to+\infty} t\int_t^{+\infty} p(s)\left(\frac{\tau(s)}{s}\right)\mathrm{d}s, \qquad H_*(0) = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t sp(s)\tau(s)\mathrm{d}s$

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where

$$r_0 = \frac{1}{2}\left(1 - \sqrt{1-4Q_*(0)}\right), \quad R_0 = \frac{1}{2}\left(1 + \sqrt{1-4H_*(0)},\right).$$

Then every proper solution to equation (1) *is oscillatory.*

Remark.

$$R_0 - r_0 = \frac{1}{2}\left(\sqrt{1-4Q_*(0)} + \sqrt{1-4H_*(0)}\right) \le \frac{1}{\mathrm{e}} \quad \text{if} \quad Q_*(0) \to \frac{1}{4}, \quad H_*(0) \to \frac{1}{4}$$

Theorem B (Koplatadze 1986). *Let*

$$\liminf_{t\to+\infty}\int_{\tau(t)}^{t}\tau(s)p(s)\mathrm{d}s > \frac{1}{\mathrm{e}}. \tag{9}$$

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Finally we assume that condition (12) holds, which also guarantee us that β is equal to 1.

Corollary 7 ($\nu = \tau$)

Let

$$Q_* \le \frac{1}{4}\,, \qquad H_* \le \frac{1}{4}\,,$$

$$\lim_{t\to+\infty} \frac{\tau(t)}{t} = 1, \tag{12}$$

and

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} sp(s)\mathrm{d}s > \frac{1}{2}\left(\sqrt{1-4Q_*} + \sqrt{1-4H_*}\right).$$

Then every proper solution to equation (1) *is oscillatory.*

$Q_* = \liminf_{t\to+\infty} t\int_t^{+\infty} p(s)\mathrm{d}s,\ H_* = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^2 p(s)\mathrm{d}s,\quad \beta = \liminf_{t\to+\infty}\left(\frac{\tau(t)}{t}\right)^{\varepsilon G_*}$

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Then every proper solution to equation (1) *is oscillatory.*

Remark. Under additional assumption (12) not only the constant $\frac{1}{\mathrm{e}}$ in Theorem B can be improved (in some cases), but it is possible to replace the lower limit by the upper limit.

Theorem B (Koplatadze 1986). *Let*

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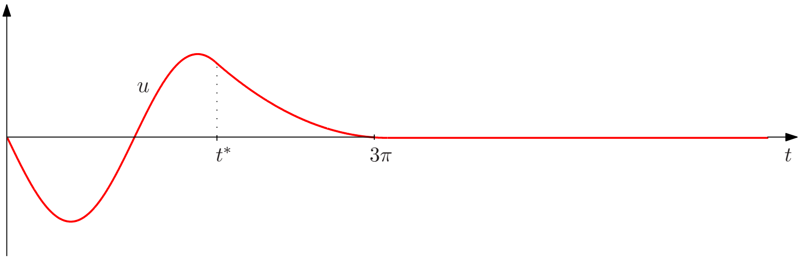
Thank you for your attention.



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