Asymptotic Representations of Solutions of Essentially Nonlinear Cyclic Systems of Ordinary Differential Equations

Elena Vladova and Oleg Shlyepakov

Odessa I.I. Mechnikov National University, Ukraine

January 20-24, 2014

We consider the system of differential equations

$$y_i' = \alpha_i p_i(t)\varphi_{i+1}(y_{i+1}) \qquad (i = \overline{1, n})^1, \tag{1}$$

where $\alpha_i \in \{-1, 1\} \quad (i = \overline{1, n})$, $p_i : [a, \omega[\to]0, +\infty[\quad (i = \overline{1, n})$ are continuous functions, $-\infty < a < \omega \leq +\infty$,
$\varphi_i : \Delta(Y_i^0) \to]0; +\infty[\quad (i = \overline{1, n})$, are once or twice continuously differentiable functions,
Y_i^0 $(i \in \{1, \ldots, n\})$ equals either 0, or $\pm\infty$, $\Delta(Y_i^0)$ $(i \in \{1, \ldots, n\})$ is one-sided neighborhood of Y_i^0.

[1] Here and in the following, all functions and parametres with subscript $n + 1$ are assumed to coincide with those with subscript 1.

Definition 1. *A positive measurable function* $\varphi_i(z)$ *defined on* $\Delta(Y_i^0)$ *is called regularly varying at* Y_i^0 *of index* σ_i *if, for each* $\lambda > 0$ *and some* $\sigma_i \in \mathbb{R}$,

$$\lim_{\substack{z \to Y_i \\ z \in \Delta\left(Y_i^0\right)}} \frac{\varphi_i(\lambda z)}{\varphi_i(z)} = \lambda^{\sigma_i} \qquad (i = \overline{1, n}). \tag{2}$$

The real number σ_i *is called the index of regular variation.*

Definition 2. *Regularly varying function* $\theta_i(z)$ *with index of regular variation equaled to 0 is called slowly varying at* Y_i^0.

Consequently, if $\varphi_i(z)$ is regularly varying of index σ_i it can be represented in the form

$$\varphi_i(z) = |z|^{\sigma_i} \theta_i(z) \qquad (i = \overline{1, n}). \tag{3}$$

Definition 3. *A positive measurable function $\varphi_i(z)$ defined on $\Delta(Y_i^0)$ is called rapidly varying at Y_i^0 if, for each $\lambda > 0$,*

$$\lim_{\substack{z \to Y_i \\ z \in \Delta\left(Y_i^0\right)}} \frac{\varphi_i(\lambda z)}{\varphi_i(z)} = \lambda^\rho \qquad (i = \overline{1, n}), \tag{4}$$

where $\rho = +\infty$, or $\rho = -\infty$.

For each $i \in \{1, \dots, n\}$ $\varphi_i(z)$ satisfies one of the following limit-relations:

- for slowly varying functions:

$$\lim_{\substack{z \to Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{z\varphi_i'(z)}{\varphi_i(z)} = \sigma_i = 0, \tag{5}$$

- for regularly varying functions:

$$\lim_{\substack{z \to Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{\varphi_i''(z)\varphi_i(z)}{\left[\varphi_i'(z)\right]^2} = \gamma_i \neq 1 \;\Rightarrow\; \lim_{\substack{z \to Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{z\varphi_i'(z)}{\varphi_i(z)} = \sigma_i = \frac{1}{1-\gamma_i} \neq 0, \tag{6}$$

-for rapidly varying functions:

$$\lim_{\substack{z \to Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{\varphi_i''(z)\varphi_i(z)}{\left[\varphi_i'(z)\right]^2} = \gamma_i = 1 \quad\Rightarrow\quad \lim_{\substack{z \to Y_i^0 \\ z \in \Delta(Y_i^0)}} \frac{z\varphi_i'(z)}{\varphi_i(z)} = \infty, \tag{7}$$

$$\varphi_i'(z) \neq 0 \quad \varphi_i(z) \to \Phi_i^0 \in \{0, +\infty\} \text{ when } z \to Y_i^0, \;\; z \in \Delta(Y_i^0).$$

$\varphi_i(z)$ $(i = \overline{1,n})$ are slowly and regularly varying.

Definition 4. *Solution* $(y_i)_{i=1}^n$ *of the system (1) is called* $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_n)$*-solution, if it is defined on the interval* $[t_0,\omega[\subset [a,\omega[$ *and satisfies following conditions*

$$y_i(t) \in \Delta(Y_i^0) \quad \textit{while} \quad t \in [t_0,\omega[, \quad \lim_{t\uparrow\omega} y_i(t) = Y_i^0,$$

$$\lim_{t\uparrow\omega} \lambda_i(t) = \Lambda_i, \quad \textit{where } \lambda_i(t) = \frac{y_i(t)y'_{i+1}(t)}{y'_i(t)y_{i+1}(t)} \ (i = \overline{1,n}). \tag{8}$$

$$\prod_{i=1}^n \lambda_i(t) = 1.$$

1) $\Lambda_1,\dots,\Lambda_n \in \mathbb{R}\setminus\{0\}$ and $\prod_{i=1}^n \Lambda_i = 1$,

2) among $\Lambda_1,\dots,\Lambda_n$ there are some equal to 0, and, therefore, equal to $\pm\infty$.

General case: $\Lambda_1, \dots, \Lambda_n \in \mathbb{R}\backslash\{0\}$.

$$\prod_{i=1}^{n} \Lambda_i = 1 \quad \prod_{i=1}^{n} \sigma_i \neq 1$$

$$\mathfrak{I} = \{i \in \{1, \dots, n\} \ : \ 1 - \Lambda_i \sigma_{i+1} \neq 0\}, \ \bar{\mathfrak{I}} = \{1, ..., n\} \backslash \mathfrak{I}, \ l = \min \mathfrak{I}.$$

$$\mu_i = \begin{cases} 1, & \text{as } Y_i^0 = +\infty, \text{ or } Y_i^0 = 0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of } 0, \\ -1, & \text{as } Y_i^0 = -\infty, \text{ or } Y_i^0 = 0 \text{ и } \Delta(Y_i^0) \text{ is left neighborhood of } 0, \end{cases}$$

$$I_i(t) = \begin{cases} \int\limits_{A_i}^{t} p_i(\tau)\, d\tau & \text{for } i \in \mathfrak{I}, \\ \int\limits_{A_i}^{t} I_l(\tau) p_i(\tau)\, d\tau & \text{for } i \in \bar{\mathfrak{I}}, \end{cases} \qquad \beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{if } i \in \mathfrak{I}, \\ \frac{\beta_l}{\prod_{k=l}^{i-1} \Lambda_k}, & \text{if } i \in \{l+1, ..., n\} \backslash \mathfrak{I}, \\ \beta_l \prod_{k=i}^{l-1} \Lambda_k, & \text{if } i \in \{1, ..., l-1\} \backslash \mathfrak{I}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such a way that corresponding integral I_i aims either to zero, or to ∞ as $t \uparrow \omega$.

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \qquad (i = 1, \dots, n).$$

Theorem 1.

Let $\Lambda_i \in \mathbb{R} \setminus \{0\} \quad (i = \overline{1,n})$ *and* $I = \min \mathfrak{I}$. *Then for the existence of* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ *- solutions of (1) it is necessary and, if algebraic equation*

$$\prod_{i=1}^{n} \left(\prod_{j=1}^{i-1} \Lambda_j + \nu \right) - \prod_{i=1}^{n} \left(\sigma_i \prod_{j=1}^{i-1} \Lambda_j \right) = 0 \tag{9}$$

does not have roots with zero real part, it is also sufficient that for each $i \in \{1, \dots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied

$$A_i^* \beta_i > 0 \quad \textit{if} \quad Y_i^0 = \pm\infty, \quad A_i^* \beta_i < 0 \quad \textit{if} \quad Y_i^0 = 0,$$

$$\operatorname{sign}\left[\alpha_i A_i^* \beta_i\right] = \mu_i.$$

Moreover, components of each solution of that type admit following asymptotic representation when $t \uparrow \omega$

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i I_i(t)[1 + o(1)], \quad if \quad i \in \mathfrak{I},$$

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i \frac{I_i(t)}{I_l(t)}[1 + o(1)], \quad if \quad i \in \bar{\mathfrak{I}},$$

and there exists the whole k*- parametric family of these solutions if there are* k *positive roots (including multiple roots) among the solutions of (9) with signs of real parts different from those of the number* $A_l^* \beta_l$*.*

Remark 1.

Algebraic equation (9) obviously does not have roots with zero real part, if $\prod_{i=1}^{n} |\sigma_i| < 1$.

Definition 5. *We define that function* θ_k $(k \in \{1, \dots, n\})$ *satisfies the condition* **S**, *if for any continuously differentiable function* $l : \Delta(Y_k^0) \longrightarrow]0, +\infty[$ *with the property*

$$\lim_{\substack{z \to Y_k^0 \\ z \in \Delta(Y_k^0)}} \frac{z\, l'(z)}{l(z)} = 0,$$

the function θ_k *admits the asymptotic representation*

$$\theta_k(zl(z)) = \theta(z)[1 + o(1)] \qquad as \quad z \to Y_k^0 \quad (z \in \Delta(Y_k^0)) \quad (1.9)$$

Theorem 2.

Let $\Lambda_i \in \mathbb{R} \setminus \{0\} \quad (i = \overline{1,n})$ *and* $l = \min \mathfrak{I}$. *Moreover, let all functions* θ_k $(k = \overline{1,n})$ *satisfy* **S**-*condition. Then each* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$-*solution (if it exists) of the system* (1) *admits for* $t \uparrow \omega$ *asymptotic representations*

$$y_i(t) = \mu_i \prod_{k=1}^{n} \left| Q_k(t)\theta_{k+1}\left(\mu_{k+1} \left|I_{k+1}(t)\right|^{\frac{1}{\beta_{k+1}}}\right)\right|^{\rho_{ik}} [1 + o(1)] \quad (i = \overline{1,n}),$$

where

$$Q_k(t) = \begin{cases} \alpha_k \beta_k I_k(t), & \text{if } k \in \mathfrak{I}, \\ \alpha_k \beta_k \dfrac{I_k(t)}{I_l(t)}, & \text{if } k \in \overline{\mathfrak{I}}, \end{cases} \qquad \rho_{ik} = \begin{cases} \dfrac{\prod\limits_{j=i+1}^{n} \sigma_j \prod\limits_{j=1}^{k} \sigma_j}{1 - \prod\limits_{j=1}^{n} \sigma_j} & \text{if } k = \overline{1, i-1}, \\ \dfrac{\prod\limits_{j=i+1}^{k} \sigma_j}{1 - \prod\limits_{j=1}^{n} \sigma_j} & \text{if } k = \overline{i, n}. \end{cases}$$

Special case: $\Lambda_i \in \mathbb{R}$ $(i = \overline{1, n-1})$ and $\Lambda_n = \pm\infty$.

$$\mathfrak{I} = \{i \in \{1, \ldots, n-1\} \,:\, 1 - \Lambda_i \sigma_{i+1} \neq 0\}, \quad \overline{\mathfrak{I}} = \{1, \ldots, n-1\} \setminus \mathfrak{I}, \quad m = \max\{i \in \mathfrak{I} : \Lambda_i = 0\}.$$

$$I_i(t) = \begin{cases} \int\limits_{A_i}^{t} p_i(\tau)\, d\tau & \text{if} \quad i \in \mathfrak{I}, \\ \int\limits_{A_i}^{t} I_{i+1}(\tau) p_i(\tau)\, d\tau & \text{if} \quad i \in \overline{\mathfrak{I}}, \end{cases} \qquad Q_i(t) = \begin{cases} \alpha_i \beta_i I_i(t) & \text{при} \quad i \in \mathfrak{I}, \\ \frac{\alpha_i \beta_i I_i(t)}{I_{i+1}(t)} & \text{при} \quad i \in \overline{\mathfrak{I}}. \end{cases}$$

$$\beta_i = \begin{cases} 1 - \Lambda_i \sigma_{i+1}, & \text{if} \quad i \in \mathfrak{I}, \\ \beta_{i+1} \Lambda_i, & \text{if} \quad i \in \overline{\mathfrak{I}}, \end{cases} \qquad M_j = \left(\prod_{i=j}^{n-1} \Lambda_i \right)^{-1} \quad (j = \overline{m+1, n-1}).$$

$$q(t) = \theta_1 \left(\mu_1 |I_1(t)|^{\frac{1}{\beta_1}} \right) |Q_{n-1}(t)|^{\prod\limits_{k=1}^{n-1} \sigma_k} \prod_{k=1}^{n-2} \left| Q_k(t) \theta_{k+1} \left(\mu_{k+1} |I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}} \right) \right|^{\prod\limits_{i=1}^{k} \sigma_i},$$

$$I_n = \int\limits_{A_n}^{t} p_n(\tau) q(\tau) d\tau, \qquad Q_n(t) = \alpha_n \beta_n I_n(t), \qquad \beta_n = 1 - \prod_{k=1}^{n} \sigma_k.$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such a way that corresponding integral I_i aims either to zero, or to ∞ as $t \uparrow \omega$.

Theorem 3.

Let $\Lambda_i \in \mathbb{R} \quad (i = \overline{1, n-1})$ *include those equal zero, and* $n-1 \in \mathfrak{I}$. *Let also functions* $\theta_k \ (k = \overline{1, n-1})$ *satisfy* **S** *- condition. Then for the existence of* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$ *- solutions of* (1)*it is necessary and, if algebraic equation*

$$(1+\lambda) \prod_{j=m+1}^{n-1} (M_j + \lambda) = \frac{\prod\limits_{j=1}^{n} \sigma_j}{\prod\limits_{j=1}^{n} \sigma_j - 1} \left(\sum_{k=m}^{n-1} \prod_{j=m+1}^{k} (M_j + \lambda) \prod_{s=k+2}^{n-1} M_s \right) \lambda \qquad (10)$$

does not have roots with zero real part, it is also sufficient that

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i} \quad (i = \overline{1, n-1})$$

and for each $i \in \{1, \dots, n\}$ *following conditions be satisfied*

$$A_i^* \beta_i > 0 \quad \text{if} \quad Y_i^0 = \pm\infty, \quad A_i^* \beta_i < 0 \quad \text{if} \quad Y_i^0 = 0,$$

$$\operatorname{sign} [\alpha_i A_i^* \beta_i] = \mu_i.$$

Moreover, components of each solution of that type admit following asymptotic representation when $t \uparrow \omega$

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))} = Q_i(t)[1 + o(1)] \quad (i = \overline{1, n-1}),$$

$$\frac{y_n(t)}{[\varphi_n(y_n(t))]^{\prod\limits_{i=1}^{n-1} \sigma_i}} = Q_n(t)[1 + o(1)],$$

and there exists the whole k- parametric family of these solutions if there are k positive numbers among

$$\gamma_i = \begin{cases} \beta_i A_i^* & \text{if} \quad i \in \mathfrak{I} \setminus \{m+1, \dots, n-1\}, \\ \beta_i A_i^* A_{i+1}^* & \text{if} \quad i \in \bar{\mathfrak{I}} \setminus \{m+1, \dots, n-1\}, \\ A_n^* \left(\prod\limits_{j=1}^{n-1} \sigma_j - 1 \right) \mathrm{Re}\, \lambda_{i-m}^0 & \text{if} \quad i \in \{m+1, \dots, n\}, \end{cases}$$

where λ_j^0 $(j = \overline{1, n-m})$ *are roots of the algebraic equation* (10) *(along with multiple).*

Theorem 4.

Let $\Lambda_i \in \mathbb{R}$ $\quad (i = \overline{1, n-1})$ include those equal zero, and $n-1 \in \mathfrak{I}$. Moreover, let all functions θ_k $(k = \overline{1, n})$ satisfy **S***-condition. Then each $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$-solution (if it exists) of the system* (1) *admits for $t \uparrow \omega$ asymptotic representations*

$$y_i(t) = \mu_i \left(\prod_{k=i}^{n-1} \left| Q_k(t) \theta_{k+1} \left(\mu_{k+1} \left| I_{k+1}(t) \right|^{\frac{1}{\beta_{k+1}}} \right) \right|^{\prod\limits_{j=i+1}^{k} \sigma_j} \right) \times$$

$$\times \left| Q_n(t) \left[\theta_n \left(\mu_n |I_n|^{\frac{1}{\beta_n}} \right) \right]^{\prod\limits_{j=1}^{n-1} \sigma_j} \right|^{\frac{\prod\limits_{j=i+1}^{n} \sigma_j}{1 - \prod\limits_{j=1}^{n} \sigma_j}} [1 + o(1)] \quad (i = \overline{1, n}).$$

$\varphi_i(z)$ $(i = \overline{1, n})$ are rapidly and regularly varying.

Definition 6. *Solution $(y_i)_{i=1}^n$ of the system (1), defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_n)$-solution $(\Lambda_1, \ldots, \Lambda_n \in \mathbb{R} \setminus \{0\})$, if functions $u_i(t) = \varphi_i(y_i(t))$ satisfy the following conditions:*

$$\lim_{t \uparrow \omega} u_i(t) = \Phi_i^0, \quad \lim_{t \uparrow \omega} \frac{u_i(t) u'_{i+1}(t)}{u'_i(t) u_{i+1}(t)} = \Lambda_i \quad (i = \overline{1, n}).$$

From the definition of $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_n)$-solution follows:

$$\prod_{i=1}^{n} L_i = 1.$$

We have at least one rapidly varying function, therefore:

$$\prod_{i=1}^{n} (1 - \gamma_i) = 0 \neq 1.$$

$$\mathfrak{I} = \{i \in \{1, \dots, n\} \,:\, 1 - \Lambda_i - \gamma_i \neq 0\}, \quad \bar{\mathfrak{I}} = \{1, \dots, n\} \setminus \mathfrak{I}, \; l = \min \mathfrak{I},$$

$$I_i(t) = \begin{cases} \int\limits_{A_i}^{t} p_i(\tau)\, d\tau & \text{if } i \in \mathfrak{I}, \\ \int\limits_{A_i}^{t} I_l(\tau) p_i(\tau)\, d\tau & \text{if } i \in \bar{\mathfrak{I}}, \end{cases}$$

$$\beta_i = \begin{cases} 1 - \Lambda_i - \gamma_i, & \text{if } i \in \mathfrak{I}, \\ \frac{\beta_l}{\prod_{k=l}^{i-1} \Lambda_k}, & \text{if } i \in \{l+1, \dots, n\} \setminus \mathfrak{I}, \\ \beta_l \prod_{k=i}^{l-1} \Lambda_k, & \text{if } i \in \{1, \dots, l-1\} \setminus \mathfrak{I}, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such a way that corresponding integral I_i aims either to zero, or to ∞ when $t \uparrow \omega$.

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \qquad (i = 1, \dots, n).$$

Theorem 5.

Let $\Lambda_i \in \mathbb{R} \setminus \{0\} \quad (i = \overline{1, n})$ *and* $I = \min \mathfrak{I}$. *Then for the existence of* $\mathcal{P}_\omega(\Lambda_1, \ldots, \Lambda_n)$ *- solutions of (1) it is necessary and, if algebraic equation*

$$\prod_{i=1}^{n} \left((1 - \gamma_i) \prod_{j=1}^{i-1} \Lambda_j + \nu \right) - \prod_{i=1}^{n} \prod_{j=1}^{i-1} \Lambda_j = 0 \tag{11}$$

does not have roots with zero real part, it is also sufficient that for each $i \in \{1, \ldots, n\}$

$$\lim_{t \uparrow \omega} \frac{I_i(t) I'_{i+1}(t)}{I'_i(t) I_{i+1}(t)} = \Lambda_i \frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied

$$A_i^* \beta_i > 0 \quad when \quad \Phi_i^0 = +\infty, \quad A_i^* \beta_i < 0 \quad when \quad \Phi_i^0 = 0,$$

$$\text{sign} \left[\alpha_i A_i^* \beta_i \right] = \text{sign} \varphi'_i(z).$$

Moreover, components of each solution of that type admit following asymptotic representation when $t \uparrow \omega$

$$\frac{\varphi_i(y_i(t))}{\varphi_i'(y_i(t))\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i I_i(t)[1 + o(1)], \quad \text{if} \quad i \in \mathfrak{I},$$

$$\frac{\varphi_i(y_i(t))}{\varphi_i'(y_i(t))\varphi_{i+1}(y_{i+1}(t))} = \alpha_i \beta_i \frac{I_i(t)}{I_l(t)}[1 + o(1)], \quad \text{if} \quad i \in \bar{\mathfrak{I}},$$

and there exists the whole k- parametric family of these solutions if there are k positive roots (including multiple roots) among the solutions of (11), with signs of real parts different from those of the number $A_l^* \beta_l$.

Theorem 6.

Let $\Lambda_i \in \mathbb{R} \setminus \{0\} \quad (i = \overline{1,n})$ *and* $l = \min \mathfrak{I}$*. Moreover, let all functions* $\theta_k(z) = \varphi_i'\left(\varphi_i^{-1}(z)\right)|z|^{-\gamma_k}$ $(k = \overline{1,n})$ *satisfy* **S***-condition. Then each* $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_n)$*-solution (if it exists) of the system* (1) *admits for* $t \uparrow \omega$ *asymptotic representations*

$$\varphi_i\left(y_i(t)\right) = \prod_{k=1}^{n} \left| Q_k(t)\theta_k\left(|I_k(t)|^{\frac{1}{\beta_k}}\right)\right|^{\delta_{ik}} [1 + o(1)] \quad (i = \overline{1,n}), \tag{12}$$

where

$$Q_k(t) = \begin{cases} \alpha_k\beta_k I_k(t), & \text{if } k \in \mathfrak{I}, \\ \alpha_k\beta_k \dfrac{I_k(t)}{I_l(t)}, & \text{if } k \in \overline{\mathfrak{I}}, \end{cases}$$

$$\delta_{ik} = \begin{cases} -\prod\limits_{j=k+1}^{i-1}(1-\gamma_j) & \text{if } k = \overline{1, i-1}, \\ -\prod\limits_{j=k+1}^{n}(1-\gamma_j)\prod\limits_{j=1}^{i-1}(1-\gamma_j) & \text{if } k = \overline{i, n}. \end{cases}$$

Remark 2.

Note that if for some $i \in \{1, ..., n\}$ $\gamma_{i-1} = 1$, then $\delta_{ik} = 0$, when $k \neq i-1$ and $\delta_{i,i-1} = -1$. Therefore, asymptotic representations (12) can be written in the form:

$$\varphi_i(y_i(t)) = \left| Q_{i-1}(t)\theta_{i-1}\left(|I_{i-1}(t)|^{\frac{1}{\beta_{i-1}}}\right)\right|^{-1}[1+o(1)] \quad \text{when } t \uparrow \omega.$$

Application of the results.

$$u^{(n)} = \alpha_0 p(t)\varphi_0(u), \tag{13}$$

$$u'' = \alpha_0 p(t)\varphi_0(u)\varphi_1(u'), \tag{14}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$ is continuous function,
$\varphi_0 : \Delta(U_0^0) \rightarrow]0; +\infty[$ is once or twice continuously differentiable function which satisfies condition (5), or (6), or (7),
$\varphi_1 : \Delta(U_1^0) \rightarrow]0; +\infty[$ is continuously differentiable function which satisfies condition (5) or (6),
U_i^0 $(i \in \{0, 1\})$ equals either 0, or $\pm\infty$, $\Delta(U_i^0)$ $(i \in \{0, 1\})$ is one-sided neighborhood of U_i^0.

Definition 7. **(φ_0 is slowly or regularly varying)** *Solution u of the equation (13) or (14) is called $P^0_\omega(\lambda_0)$-solution ($\lambda_0 \in \mathbb{R} \bigcup \{\pm\infty\}$), if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies following conditions*

$$\lim_{t\uparrow\omega} u(t) = U^0_0, \quad \lim_{t\uparrow\omega} u^{(k)}(t) = U^0_k \in \{0, \pm\infty\} \ (k = 1, \dots, n-1),$$

$$\lim_{t\uparrow\omega} \frac{[u^{(n-1)}(t)]^2}{u^{(n)}(t)u^{(n-2)}(t)} = \lambda_0.$$

Definition 8. **(φ_0 is rapidly varying)** *Solution u of the equation (13) or (14) is called $P^\infty_\omega(\lambda_0)$-solution ($\lambda_0 \in \mathbb{R} \setminus \{0\}$), if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies following conditions*

$$\lim_{t\uparrow\omega} \varphi(u(t)) = \Phi^0, \quad \lim_{t\uparrow\omega} u^{(k)}(t) = U^0_k \in \{0, \pm\infty\} \ (k = 1, \dots, n-1)$$

$$\lim_{t\uparrow\omega} \frac{\varphi_0(u(t))}{\varphi_0'(u(t))} \frac{u''(t)}{[u'(t)]^2} = \lambda_0 \quad \text{and} \quad \lim_{t\uparrow\omega} \frac{[u^{(n-1)}(t)]^2}{u^{(n)}(t)u^{(n-2)}(t)} \text{ exists.}$$