ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH REGULARLY VARYING NONLINEARITIES

V.M. Evtukhov

Odessa I.I.Mechnikov National University. Ukraine

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Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$- is a continuous function, $\varphi_j : \triangle_{Y_j} \longrightarrow]0, +\infty[$ $(j = \overline{0, n-1})$- continuous regularly varying at $y^{(j)} \longrightarrow Y_j$ functions of orders σ_j, $-\infty < a < \omega \leq +\infty$, $\triangle_{Y_j}$ - one-sided neighborhood Y_j, Y_j equals either 0, or $\pm\infty$.

From definition of regularly varying function it follows that

$$\varphi_j\left(y^{(j)}\right) = \left|y^{(j)}\right|^{\sigma_j} L_j\left(y^{(j)}\right) \qquad (j = \overline{0, n-1}),$$

where $L_j : \Delta_{Y_j} \longrightarrow]0, +\infty[$ $(j = \overline{0, n-1})$ are continuous and slowly varying at $y^j \to Y_j$ functions, i.a. such that conditions

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta_{Y_j}}} \frac{L_j\left(\lambda y^{(j)}\right)}{L_j\left(y^{(j)}\right)} = 1 \qquad (j = \overline{0, n-1})$$

are satisfied for each $\lambda > 0$.
For example, the following functions are slowly varying as $y \longrightarrow Y_0$ (Y_0 is either 0, or $\pm\infty$):

$$\ln^k |y|, \quad \ln^m |\ln|y|| \quad (k, m \in \mathbb{R} \setminus \{0\}),$$

$$e^{(|\ln|y||)^\alpha} \quad (0 < \alpha < 1), \quad e^{\frac{\ln|y|}{\ln|ln|y||}},$$

they have a nonzero finite limit as $y \longrightarrow Y_0$.

At study of the equation (1) we will assume that the numbers defined by

$$\nu_j = \begin{cases} 1, & \text{if either} \quad Y_j = +\infty, \quad \text{or} \quad Y_j = 0 \quad \text{and} \quad \Delta_{Y_j} - \text{right neighborhood of } 0, \\ -1, & \text{if either} \quad Y_j = -\infty, \quad \text{or} \quad Y_j = 0 \quad \text{and} \quad \Delta_{Y_j} - \text{left neighborhood of } 0, \end{cases}$$

such that

$$\nu_j \nu_{j+1} > 0 \quad \text{as} \quad Y_j = \pm\infty \quad \text{and} \quad \nu_j \nu_{j+1} < 0 \quad \text{as} \quad Y_j = 0 \quad (j = \overline{0, n-2}). \tag{2}$$

These conditions are necessary for existence at the equation (1) solutions defined in the left neighborhood ω, each of which satisfies to conditions

$$y^{(j)}(t) \in \Delta_{Y_j} \qquad \text{as } t \in [t_0, \omega[\;, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \quad (j = \overline{0, n-1}). \tag{3}$$

Among set of all solutions of the equation (1) defined in the left neighborhood ω they represent the greatest interest as each of the remaining has only one of representations

$$y(t) = \pi_\omega^{k-1}(t)[c_{k-1} + o(1)] \quad (k = \overline{1, n}) \quad t \uparrow \omega, \tag{4}$$

where c_{k-1} $(k = \overline{1, n})$ are real constants distinct from zero,

$$\pi_\omega(t) = \begin{cases} t, & \text{if} \quad \omega = +\infty, \\ t - \omega, & \text{if} \quad \omega < +\infty. \end{cases}$$

The problem on existence at the equation (1) solutions with representations (4) as a whole can be solved with use of known results and research methods, for example, at $\omega = +\infty$ with use of theorems of I.T.Kiguradze.
As to solutions with properties (3) for them a priori it is not had concrete asymptotic representations. Therefore first of all there is a necessity of allocation from their set of a class of solutions for which such representations can be established. One of such enough wide classes has been introduced in my papers devoted generalized differential equation of type of Emden-Fowler

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} |y^{(j)}|^{\sigma_j}.$$

Definition 1.

Solution y of the equation (1) is called $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$*- solution, where* $-\infty \le \lambda_0 \le +\infty$*, if it is defined on an interval* $[t_0, \omega[\subset [a, \omega[$*, satisfies to conditions (3) and such that*

$$\lim_{t\uparrow\omega} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t)y^{(n)}(t)} = \lambda_0.$$

If y is a solution with properties (3) differential equation (1) and functions $\ln|y^{(n-1)}(t)|$ and $\ln|\pi_\omega(t)|$ comparable an order 1 at $t \uparrow \omega$ it is easy to prove that the given solution is a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution at some value λ_0, depending on value of a limit

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y^{(n)}(t)}{y^{(n-1)}(t)}.$$

If coefficient p the equations (1) is regularly varying function at $t \uparrow \omega$ it is possible to show that each regularly varying solution with properties (3) these equations is a $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solution at some value λ_0.

$P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$- solutions possess different asymptotic properties as $t \uparrow \omega$ depending on values λ_0, namely,
when $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\right\}$ – principal case,
when $\lambda_0 \in \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1, \pm\infty\right\}$ - special (worst) cases.

In the present report for the equation (1) will be presented at each of possible values λ_0 the established results on existence and asymptotic behaviour of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$- solutions.

Principal case: $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \dots, \frac{n-2}{n-1}, 1\right\}$.

$$a_{0i} = (n-i)\lambda_0 - (n-i-1) \quad (i = 1, \dots, n) \quad \text{при} \quad \lambda_0 \in \mathbb{R},$$

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \;\; \mu_n = \sum_{j=0}^{n-2} \sigma_j (n-j-1), \;\; C = \prod_{j=0}^{n-2} \left| \frac{(\lambda_0 - 1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_j},$$

$$J_n(t) = \int_{A_n}^{t} p(\tau) |\pi_\omega(\tau)|^{\mu_n} \, d\tau,$$

where an integration limit A_n gets out equal a if at this value the integral aspires to $+\infty$ as $t \uparrow \omega$, and equal ω if at such value of a limit of integration the integral aspires to zero as $t \uparrow \omega$.

$$y^{(k-1)}(t) \sim \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-k}}{\prod\limits_{i=k}^{n-1} a_{0i}} y^{(n-1)}(t) \quad (k = 1, \dots, n-1)$$

Theorem 1.

Let $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1\right\}$ *and* $\gamma_0 \neq 0$. *Then for existence of* $P_\omega(Y_0, \ldots, Y_{n-1}, \lambda_0)$*-solutions of equation* (1) *it is necessary and if algebraic equation*

$$\sum_{j=0}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} a_{0\,i} \prod_{i=1}^{j} (\, a_{0\,i} + \rho\,) = (\,1 + \rho\,) \prod_{i=1}^{n-1} (\, a_{0\,i} + \rho\,) \tag{5}$$

does not have roots with zero real part, is sufficiently that inequality (2), *inequalities*

$$\nu_{0j}\nu_{0j+1}a_{0j+1}(\lambda_0 - 1)\pi_\omega(t) > 0 \quad (j = \overline{0, n-2}), \quad \alpha_0\nu_{n-1}\gamma_0 J_n(t) > 0 \quad at \quad t \in]a, \omega[$$

and condition

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_n'(t)}{J_n(t)} = \frac{\gamma_0}{\lambda_0 - 1}$$

are satisfied.

Moreover, for each such solution as $t \uparrow \omega$ following asymptotic representations are valid

$$y^{(j)}(t) = \frac{[(\lambda_0 - 1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t)[1 + o(1)] \quad (j = 0, 1, \dots, n-2),$$

$$\frac{\left|y^{(n-1)}(t)\right|^{\gamma_0}}{\prod\limits_{j=0}^{n-1} L_j\left(\frac{[(\lambda_0-1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t)\right)} = \alpha_0 \nu_{n-1}\gamma_0 C J_n(t)[1 + o(1)],$$

where $L_j\left(y^{(j)}\right) = |y^{(j)}|^{-\sigma_j}\varphi_{sj}\left(y^{(j)}\right) \quad (j = \overline{0, n-1})$. Furthermore, there exists an m-parameter family of such solutions if , among the roots of equation (5), there are m roots (with regard of multiplicities) with the real part having the same sign as the function $(1 - \lambda_0)\pi_\omega(t)$.

Remark 1.

The algebraic equation (5) obviously has no roots with a zero real part, if

$$\sum_{j=0}^{n-2} |\sigma_j| < |\sigma_{n-1} - 1|$$

.

Under certain additional assumptions the asymptotic representations in the theorem 1 can be written down in an explicit form.

Definition 2

We call the slowly varying as $z \to Z_0$ *function* $L : \Delta_{Z_0} \to]0, +\infty[$, *where* Z_0 *is either 0, or* $\pm\infty$, Δ_{Z_0} *is one-sided neighborhood of* Z_0, *satisfies the Condition* S_0, *if representation*

$$L\left(\nu e^{[1+o(1)]\ln|z|}\right) = L(z)[1+o(1)] \quad \textit{при} \quad z \to Z_0 \quad (z \in \Delta_{Z_0}),$$

where $\nu = \operatorname{sign} z$, *takes place.*

Theorem 2.

Let the conditions of theorem 1 be satisfied and the functions L_j $(j=\overline{0,n-1})$ *satisfy Condition* S_0. *Then each* $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$*- solution of differential equation (1) admits the following asymptotic representations as* $t\uparrow\omega$

$$y^{(j)}(t)\sim\frac{\nu_{n-1}[(\lambda_0-1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\left|\gamma_0 CJ_n(t)\prod_{i=0}^{n-1}L_i\left(\nu_i|\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_0-1}}\right)\right|^{\frac{1}{\gamma_0}}\quad (j=\overline{0,n-1}).$$

Special cases: $\lambda_0 = 1$ and $\lambda_0 = \pm\infty$.

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-2} \sigma_j (n - j - 1),$$

$$J_0(t) = \int_{A_0}^{t} p(s)\, ds, \quad J_{00}(t) = \int_{A_{00}}^{t} J_0(s)\, ds,$$

$$\widetilde{J}_n(t) = \int_{A_n}^{t} p(s) |\pi_\omega(s)|^{\mu_n} \prod_{j=0}^{n-2} L_j \left(\nu_j |\pi_\omega(s)|^{n-j-1} \right) ds.$$

$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-k}}{(n-k)!}\, y^{(n-1)}(t) \quad (k = 1, \ldots, n-1),$$

$$y^{(n)}(t) = o\left(\frac{y^{(n-1)}(t)}{\pi_\omega(t)}\right);$$

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \cdots \sim \frac{y^{(n)}(t)}{y^{(n-1)}(t)}$$

Theorem 3.

Let $\gamma_0 \neq 0$. *Then for existence of* $P_\omega(Y_0, \dots, Y_{n-1}, 1)$*- solutions of equation* (1) *it is necessary and if algebraic equation*

$$(1+\rho)^n = \sum_{j=0}^{n-1} \sigma_j (1+\rho)^j \qquad (6)$$

does not have roots with zero real part, is sufficiently that inequality (2), *inequalities*

$$\alpha_0 \nu_{n-1} \gamma_0 J_0(t) > 0, \quad \nu_j \nu_{n-1} \left(\gamma_0 J_0(t)\right)^{n-j-1} > 0 \quad (j = \overline{0, n-2}) \quad \textit{при} \quad t \in]a, \omega[$$

and conditions

$$\lim_{t \uparrow \omega} \frac{p(t) J_{00}(t)}{J_0^2(t)} = 1, \quad \nu_j \lim_{t \uparrow \omega} |J_0(t)|^{\frac{1}{\gamma_0}} = Y_j \quad (j = \overline{0, n-1}).$$

are satisfied.

Moreover, for each such solution as $t \uparrow \omega$ following asymptotic representations are valid

$$y^{(j)}(t) = \left(\frac{\gamma_0 J_{00}(t)}{J_0(t)}\right)^{n-j-1} y^{(n-1)}(t)[1+o(1)] \quad (j = \overline{0, n-2}),$$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{\prod\limits_{j=0}^{n-1} L_j\left(\left(\frac{\gamma_0 J_{00}(t)}{J_0(t)}\right)^{n-j-1} y^{(n-1)}(t)\right)} = \alpha_0 \nu_{n-1} \gamma_0 J_0(t) \left|\frac{\gamma_0 J_{00}(t)}{J_{00}(t)}\right|^{\mu_n} [1+o(1)].$$

Furthermore, there exists an m-parameter family of such solutions if , among the roots of equation (6), there are m roots (with regard of multiplicities) which the real part have a sign opposite to a sign $\alpha_0 \nu_{n-1}$.

Remark 2.

The algebraic equation (6) obviously has no roots with a zero real part, if

$$\sum_{j=0}^{n-2} |\sigma_j| < |\sigma_{n-1} - 1|$$

.

Theorem 4.

Let the conditions of theorem 3 be satisfied and the functions L_j $(j = \overline{0, n-1})$ satisfy Condition S_0. Then each $P_\omega(Y_0, \dots, Y_{n-1}, 1)$- solution of differential equation (1) admits the following asymptotic representations as $t \uparrow \omega$

$$y^{(j)}(t) \sim \nu_{n-1} \left(\frac{\gamma_0 J_0(t)}{p(t)} \right)^{n-j-1} \left| \gamma_0 J_0(t) \left| \frac{\gamma_0 J_0(t)}{p(t)} \right|^{\mu_n} \prod_{j=0}^{n-1} L_j \left(\nu_j |J_0(t)|^{\frac{1}{\gamma_0}} \right) \right|^{\frac{1}{\gamma_0}} \quad (j = \overline{0, n-1}).$$

Theorem 5.

Let $\gamma_0 \neq 0$ *and the functions* L_j $(j = \overline{0, n-2})$ *satisfy Condition* S_0. *Then for existence of* $P_\omega(Y_0, \dots, Y_{n-1}, \pm\infty)$*- solutions of equation* (1) *it is necessary and sufficiently that inequality* (2), *inequalities*

$$\nu_j \nu_{n-1} \pi_\omega^{n-j-1}(t) > 0 \quad (j = \overline{0, n-2}), \qquad \alpha_0 \nu_{n-1} \gamma_0 \widetilde{J}_n(t) > 0$$

and conditions

$$\nu_j \lim_{t \uparrow \omega} |\pi_\omega(t)|^{n-j-1} = Y_j \quad (j = \overline{0, n-2}), \qquad \nu_{n-1} \lim_{t \uparrow \omega} |\widetilde{J}_n(t)|^{\frac{1}{\gamma_0}} = Y_{n-1},$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \widetilde{J}_n'(t)}{\widetilde{J}_n(t)} = 0.$$

are satisfied.

Moreover, for each such solution as $t \uparrow \omega$ *following asymptotic representations are valid*

$$y^{(j-1)}(t) \sim \frac{[\pi_\omega(t)]^{n-j}}{(n-j)!}\, y^{(n-1)}(t)[1+o(1)] \quad (j=1,\dots,n-1),$$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{L_{n-1}(y^{(n-1)}(t))} = \alpha_0 \nu_{n-1}\gamma_0 \prod_{j=0}^{n-2} \left|\frac{1}{(n-j-1)!}\right|^{\sigma_j} \widetilde{J}_n(t)[1+o(1)].$$

Furthermore, if $\omega = +\infty$ *there is n-parametrical (n-1-parametrical) family of such solutions in a case, when* $\widetilde{J}_n(t) > 0$ $(\widetilde{J}_n(t) < 0)$ *at* $t \in [a_0, \omega[$*; if* $\omega < +\infty$ *and* $\widetilde{J}_n(t) > 0$ *at* $t \in [a_0, \omega[$ *there is a one-parametrical family of such solutions.*

Theorem 6.

Let the conditions of theorem 5 be satisfied and the function L_{n-1} satisfy Condition S_0. Then each $P_\omega(Y_0, \dots, Y_{n-1}, \pm\infty)$- solution of differential equation (1) admits the following asymptotic representations as $t \uparrow \omega$

$$y^{(j-1)}(t) \sim \frac{\nu_{n-1}[\pi_\omega(t)]^{n-j}}{(n-j)!} \left| \gamma_s \prod_{j=0}^{n-2} \left| \frac{1}{(n-j-1)!} \right|^{\sigma_{sj}} \widetilde{J}_{sn}(t) L_{sn-1}\left(\nu_{n-1} |\widetilde{J}_{sn}(t)|^{\frac{1}{\gamma_s}} \right) \right|^{\frac{1}{\gamma_s}} \quad (j = \overline{1, n})$$

Special cases: $\lambda_0 = \frac{n-i-1}{n-i}$ $(i = \overline{1, n-1})$.

$$\mu_i = n - i - 1 + \sum_{j=0}^{i-2} \sigma_j (i - j - 1) - \sum_{j=i+1}^{n-1} \sigma_j (j - i) \quad (i = \overline{1, n}),$$

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \gamma_i = 1 - \sum_{j=i}^{n-1} \sigma_j \quad (i = \overline{1, n-1}),$$

$$C_i = \frac{1}{(n-i)!} \prod_{j=0}^{i-1} [(i-j-1)!]^{-\sigma_j} \prod_{j=i+1}^{n-1} [(j-i)!]^{\sigma_j} \ (i = \overline{1, n-1}),$$

$$J_i(t) = \int_{A_i}^{t} p(s) |\pi_\omega(s)|^{\mu_i} \prod_{\substack{j=0 \\ j \neq i-1}}^{n-1} L_j \left(\nu_j |\pi_\omega(s)|^{i-j-1} \right) ds \ (i = \overline{1, n}),$$

$$J_{ii}(t) = \int_{A_{ii}}^{t} |J_i(s)|^{\frac{1}{\gamma_i}} ds \ (i = \overline{1, n}).$$

$$y^{(k-1)}(t) \sim \frac{[\pi_\omega(t)]^{i-k}}{(i-k)!}\, y^{(i-1)}(t) \quad (k = 1, \dots, i-1),$$

$$y^{(i)}(t) = o\left(\frac{y^{(i-1)}(t)}{\pi_\omega(t)}\right),$$

$$y^{(k)}(t) \sim (-1)^{k-i}\frac{(k-i)!}{[\pi_\omega(t)]^{k-i}}\, y^{(i)}(t) \quad (k = i+1, \dots, n).$$

$i = n-1$

$\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)}$.

Theorem 7.

Let $i \in \{1, \dots, n-1\}$, $\gamma_0\gamma_i \neq 0$ *and functions* L_j *at all* $j \in \{0, \dots, n-1\} \setminus \{i-1\}$ *satisfy to a condition* S_0. *Then for existence at the equation* (1) $P_\omega\left(Y_0, \dots, Y_{n-1}, \frac{n-i-1}{n-i}\right)$ *- solutions (at* $i = n-1$ *for which exists finit or equal* $\pm\infty$ *a limit* $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t)y^{(n)}(t)}{y^{(n-1)}(t)}$*) is necessary, and if the algebraic equation*

$$\sum_{j=i+1}^{n-1} \frac{\sigma_j}{(j-i)!} \prod_{m=1}^{j-i} (m-\rho) + \sigma_i = \frac{1}{(n-i)!} \prod_{m=1}^{n-i} (m-\rho) \tag{7}$$

does not have roots with zero real part, is sufficiently that inequality (2), *inequalities*

$$\nu_j\nu_{j-1}(i-j)\pi_\omega(t) > 0 \ \text{ at all } j \in \{1, \dots, n-1\} \setminus \{i\}, \quad \nu_i\nu_{i-1}\gamma_0\gamma_i J_{ii}(t) > 0,$$

$$\nu_i\alpha_0(-1)^{n-i-1}\pi_\omega^{n-i-1}(t)\gamma_i J_i(t) > 0$$

and conditions

$$\nu_{j-1} \lim_{t\uparrow\omega} |\pi_\omega(t)|^{i-j} = Y_{j-1} \quad \text{при всех} \quad j \in \{1, \dots, n\} \setminus \{i\}, \quad \nu_{i-1} \lim_{t\uparrow\omega} |J_{ii}(t)|^{\frac{\gamma_i}{\gamma_0}} = Y_{i-1},$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_i'(t)}{J_i(t)} = -\gamma_i, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_{ii}'(t)}{J_{ii}(t)} = 0.$$

are satisfied.

Moreover, for each such solution as $t \uparrow \omega$ following asymptotic representations are valid

$$y^{(j-1)}(t) = \frac{[\pi_\omega(t)]^{i-j}}{(i-j)!} y^{(i-1)}(t)[1+o(1)] \quad (j = 1, \dots, i-1),$$

$$y^{(j)}(t) = (-1)^{j-i} \frac{(j-i)!}{[\pi_\omega(t)]^{j-i}} \cdot \frac{\gamma_i J'_{ii}(t)}{\gamma_0 J_{ii}(t)} y^{(i-1)}(t)[1+o(1)] \quad (j = i, \dots, n-1),$$

$$\frac{|y^{(i-1)}(t)|^{\gamma_0}}{L_{i-1}(y^{(i-1)}(t))} = |\gamma_i C_i| \left| \frac{\gamma_0}{\gamma_i} J_{ii}(t) \right|^{\gamma_i} [1+o(1)] \quad \text{при} \quad t \uparrow \omega.$$

Furthermore, such solutions in a case, when $\omega = +\infty$ exists $i + l$-parameter ($i - 1 + l$-parameter) family if $\nu_i \nu_{i-1} \gamma_0 \gamma_i > 0$ ($\nu_i \nu_{i-1} \gamma_0 \gamma_i < 0$) and in a case when $\omega < +\infty$ exist an $r + 1$-parameter (r- parameter) family if $\nu_i \nu_{i-1} \gamma_0 \gamma_i > 0$ ($\nu_i \nu_{i-1} \gamma_0 \gamma_i < 0$), where l (r)- number of roots of an equation (7) with a negative (positive) real part.

Remark 3.

The algebraic equation (7) obviously has no roots with a zero real part, if

$$\sum_{j=i}^{n-2} |\sigma_j| < |1 - \sigma_{n-1}|.$$

Theorem 8.

Let the conditions of theorem 7 be satisfied and the function L_{i-1} satisfy Condition S_0. Then each $P_\omega\left(Y_0,\dots,Y_{n-1},\frac{n-i-1}{n-i}\right)$- solution of differential equation (1) admits the following asymptotic representations as $t\uparrow\omega$

$$y^{(j-1)}(t)\sim\frac{\nu_{i-1}[\pi_\omega(t)]^{i-j}}{(i-j)!}\left|\gamma_i C_i L_{i-1}\left(\nu_{i-1}|J_{ii}(t)|^{\frac{\gamma_i}{\gamma_0}}\right)\right|^{\frac{1}{\gamma_0}}\left|\frac{\gamma_0}{\gamma_i}J_{ii}(t)\right|^{\frac{\gamma_i}{\gamma_0}}\ (j=\overline{1,i}),$$

$$y^{(j)}(t)\sim(-1)^{j-i}\frac{\nu_{i-1}(j-i)!}{[\pi_\omega(t)]^{j-i}}\cdot\frac{\gamma_i J_{ii}'(t)}{\gamma_0 J_{ii}(t)}\left|\gamma_i C_i L_{i-1}\left(\nu_{i-1}|J_{ii}(t)|^{\frac{\gamma_i}{\gamma_0}}\right)\right|^{\frac{1}{\gamma_0}}\left|\frac{\gamma_0}{\gamma_i}J_{ii}(t)\right|^{\frac{\gamma_i}{\gamma_0}}$$

$$(j=\overline{i,n-1}).$$

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