ON SPECIAL SOLUTIONS TO EMDEN—FOWLER TYPE DIFFERENTIAL EQUATIONS

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Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \qquad (1)$$

$n \geq 2$, $k > 1$,

$p(x, y_0, \dots, y_{n-1}) > 0, \quad p \in C(\mathbf{R}^n)$.

I. T. Kiguradze posed the problem on asymptotic behavior of all non-extensible solutions to this equation such that

$$\lim_{x \to x^* - 0} y(x) = \infty. \qquad (2)$$

.

He found the asymptotic formulae for such solutions to equation (1) with $n = 2$.

See [*Kiguradze I. T., Chanturia T. A.* Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluver Academic Publishers, Dordreht-Boston-London. 1993.]

The problem was completely solved for $n = 3$ and $n = 4$ (I.Astashova).

For equation (1) with $2 \leq n \leq 11$, existence was proved of an $(n-1)$-parametric family of solutions with the vertical asymptote $x = x^*$, all having the form

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0, \tag{3}$$

with $\alpha = \frac{n}{k-1}$, $C = \left(\frac{\alpha(\alpha+1)...(\alpha+n-1)}{p_0}\right)^{\frac{1}{k-1}}$,

$$p_0 = \lim p\,(x, y_0, y_1 \ldots, y_{n-1})\,,$$

$$x \to x^* - 0,\ y_0 \to \infty,\ y_1 \to \infty, \ldots, y_{n-1} \to \infty,$$

$$p_0 > 0.$$

if the following two conditions hold:

1. The continuous positive function $p(x,\, y_0,\, \ldots, y_{n-1})$ has a limit $p_0 = \mathrm{const} > 0$ as $x \to x^* - 0$, $y_0 \to \infty$, ..., $y_{n-1} \to \infty$, and for some $\gamma > 0$ it holds

$$p(x,\, y_0,\, \ldots, y_{n-1}) - p_0 = O\left(|x^* - x|^\gamma + \sum_{j=0}^{n-1} {y_j}^{-\gamma} \right).$$

2. For some $K_1 > 0$ and $\mu > 0$ in a neighborhood of x^* for rather large y_0, ..., y_{n-1}, z_0, ..., z_{n-1} it holds

$$\left| p(x,\, y_0,\, \ldots, y_{n-1}) - p(x,\, z_0,\, \ldots, z_{n-1}) \right| \leq$$

$$K_1 \max_j \left| {y_j}^{-\mu} - {z_j}^{-\mu} \right|.$$

It was also proved for the equation

$$y^{(n)} = (-1)^n |y|^k \operatorname{sign} y, \quad k > 1, \tag{4}$$

with $n = 3$ and $n = 4$ that all its Kneser solutions, i.e. solutions defined near $+\infty$ and satisfying in their domains the inequalities

$$(-1)^j y^{(j)} > 0, \quad j = 0, \dots, n-1,$$

have the form

$$y(x) = C(x - x^*)^{-\alpha} \tag{5}$$

with arbitrary x^* and the same C and α as in (3).

See

[*Astashova I. V.* Asymptotic behavior of solutions of certain nonlinear differential equations, In Reports of extended session of a seminar of the I.N.Vekua Institute of Applied Mathematics. Tbilisi. 1985. v. 1. N 3. p.9–11. (Russian)],

[*Astashova I. V.* Application of Dynamical Systems to the Study of Asymptotic Properties of Solutions to Nonlinear Higher-Order Differential Equations, Journal of Mathematical Sciences. Springer Science+Business Media. 126 (2005), no. 5, 1361-1391.]

For the equation

$$y^{(n)} = |y|^{\,k}, \;\; k > 1, \tag{6}$$

a negative answer to a conjecture posed by Kiguradze was obtained. It was proved, that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$, $1 < k < K$, such that equation (6) has a solution

$$y(x) = (x^* - x)^{-\alpha} \; h(\log{(x^* - x)}), \tag{7}$$

where h is a periodic positive nonconstant function on $\mathbf{R}$.

See [*Kozlov V. A.* On Kneser solutions of higher order nonlinear ordinary differential equations. Ark. Mat., 1999, vol. 37, no. 2, p. 305–322.]

We proved the following result

Theorem *For* $12 \le n \le 14$ *there exists* $k > 1$ *such that equation (6) has a solution* $y(x)$ *with*

$$y^{(j)}(x) = (x^* - x)^{-\alpha - j}\, h_j(\,\ln(x^* - x)\,),$$

$$j = 0, 1, \ldots, n-1,$$

where h_j *are periodic positive non-constant functions on* $\mathbf{R}$.
(for $n = 12$ with S.Vyun).

Note that the substitution $x \mapsto -x$ transforms this $y(x)$ into a Kneser solution to equation (4). And the latter solution does not match the standard form given by (5).

Reference 1. To prove this result we used method based on the Hopf Bifurcation theorem.

See [*J. E. Marsden, M. McCracken.* The Hopf bifurcation and its applications. Springer- Verlag, New York, 1976, XIII, 408 pp.]

This method does not allow to obtain the same result for $n < 12$, but this does not mean that there is no solution of the form given by (7) if $5 \leq n \leq 11$.

Theorem (Hopf). *Consider the α-parameterized dynamical system $\dot{x} = L_\alpha x + Q_\alpha(x)$ in a neighborhood of $0 \in \mathbf{R}^n$ with linear operators L_α and smooth enough functions $Q_\alpha(x) = O\left(|x|^2\right)$ as $x \to 0$. Let λ_α and $\bar{\lambda}_\alpha$ be complex conjugated eigenvalues of the operators L_α. Suppose $\mathrm{Re}\lambda_{\tilde{\alpha}} = \mathrm{Re}\bar{\lambda}_{\tilde{\alpha}} = 0$ for some $\tilde{\alpha}$ and the operator $L_{\tilde{\alpha}}$ has no other eigenvalues with zero real part.*
If $\mathrm{Re}\frac{d\lambda_\alpha}{d\alpha}\left(\tilde{\alpha}\right) \neq 0$, then there exist continuous mappings $\alpha(\varepsilon) \in \mathbf{R}$, $T(\varepsilon) \in \mathbf{R}$, and $b(\varepsilon) \in \mathbf{R}^n$ such that $\alpha(0) = \tilde{\alpha}$, $T(0) = 2\pi/\mathrm{Im}\lambda_{\tilde{\alpha}}$, $b(0) = 0,\ \ b(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, and the solutions of the tasks

$$\dot{x} = L_{\alpha(\varepsilon)} x + Q_{\alpha(\varepsilon)}(x), \qquad x(0) = b(\varepsilon)$$

are $T(\varepsilon)$-periodic and non-constant.

Reference 2. To apply the Hopf Bifurcation theorem we investigate the roots of the algebraic equation

$$\prod_{j=1}^{n}(\alpha + j) = \prod_{j=0}^{n-1}(\alpha + j + \lambda). \tag{8}$$

If this equation has a pair of pure imagine roots we can apply the Hopf Bifurcation theorem.

In order to investigate asymptotic behavior of all ultimately positive solutions to the equation

$$y^{(n)} = |y|^k \operatorname{sign} y, \tag{9}$$

we used the auxiliary variables

$$u_j = y^{(j)} y^{-\beta_j} \quad \text{with } \beta_j = 1 + \frac{j}{\alpha}, \quad j = 1, \dots, n-1,$$

and a new independent variable given by

$$t = \int_{x_0}^{x} y(\xi)^{\frac{1}{\alpha}} \, d\xi.$$

Equation (9) is transformed by this way into the system

$$\begin{cases} \dot{u}_1 = u_2 - \beta_1 u_1^2, \\ \dot{u}_j = u_{j+1} - \beta_j u_1 u_j, \quad j = 2, \dots, n-2, \\ \dot{u}_{n-1} = 1 - \beta_{n-1} u_1 u_{n-1}. \end{cases} \tag{10}$$

In the domain with $u_j > 0$, this system has a single fixed point. The related constant trajectory corresponds to the family of standard solutions to (6) given by the power function

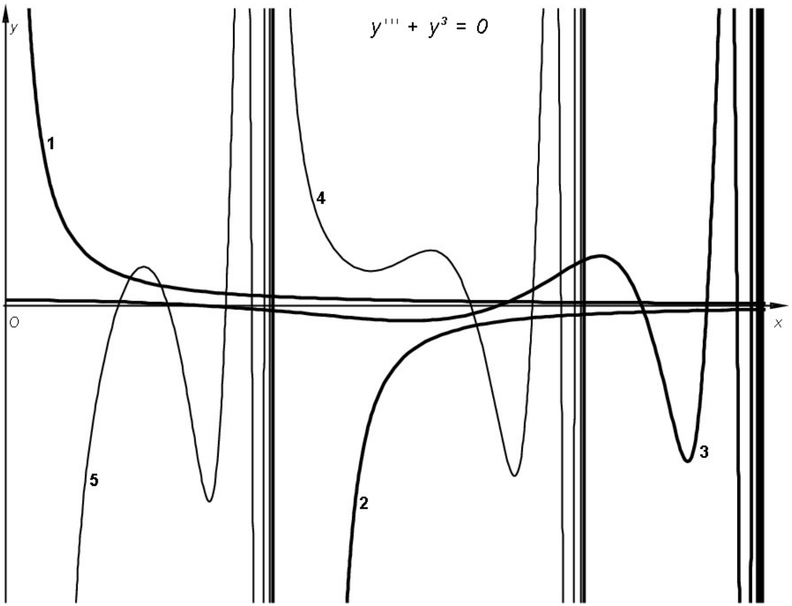
$$y(x) = C(x^* - x)^{-\alpha}, \quad x < x^*.$$

For $2 \le n \le 4$, it is proven that all "positive" trajectories of system (10) tend to the fixed point. Accordingly, all ultimately positive solutions to (6) have vertical asymptotes with power-law asymptotic behavior, i. e.

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)) \text{ as } x \to x^*.$$

The same result for equation (1) is proved using the same variables u_j and t, but with a more complicated system similar to (10).

We can consider $\{u_j\}$ as the set of coordinate functions producing a chart on a compact "phase"manifold. Together with other similar charts given by similar formulae like
$v_j = y^{(j)}\ |y'|^{-\beta_j/\beta_1}\ ,\ j = 0, 2, \ldots, n-1$, we can cover the whole "phase"manifold and globally define a dynamical system expanding system (10). This allows to obtain all possible types of behavior for solutions to equations (4) and (9) with $n = 3$ and $n = 4$. See an illustration for $n = 3$:

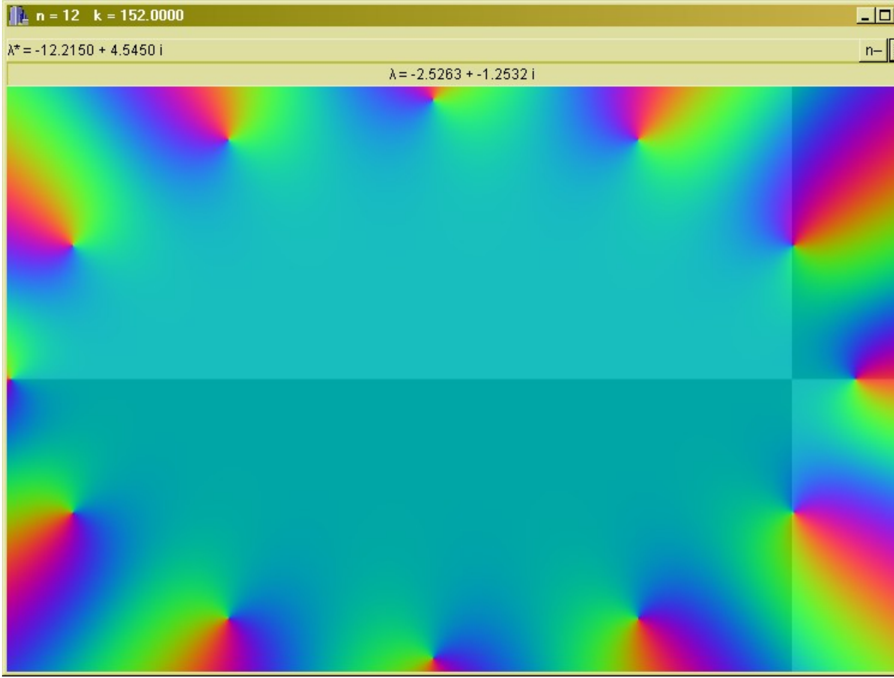

$y''' + y^3 = 0$
y
O
x
1
2
3
4
5

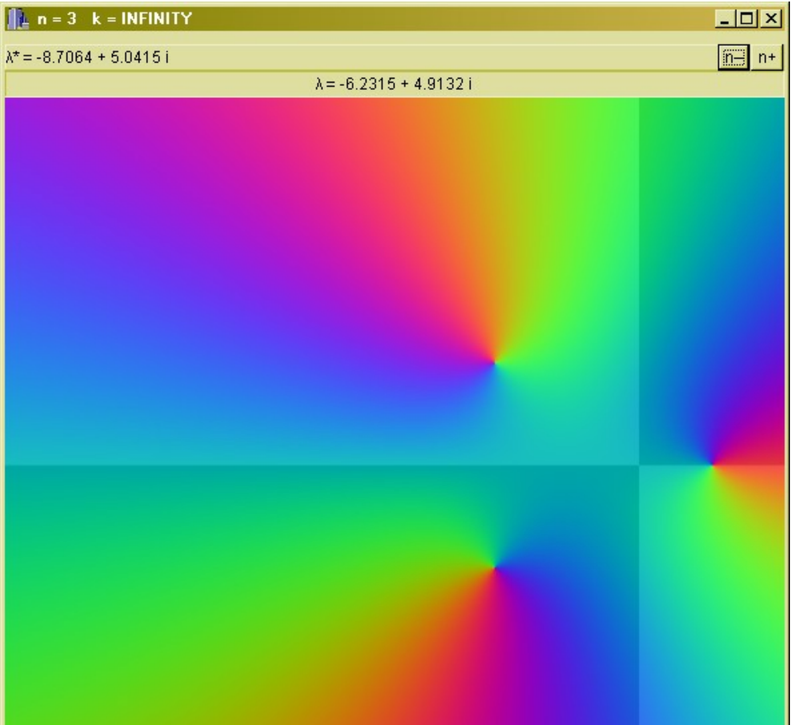
For equation (1) of higher orders, existence of $(n-1)$-parametrical family of solutions with the same power-low asymptotic behavior can be proven by using the substitution $x = e^t, \ y = (C+v)\, e^{-\alpha t}$ transforming equation (1) into the system

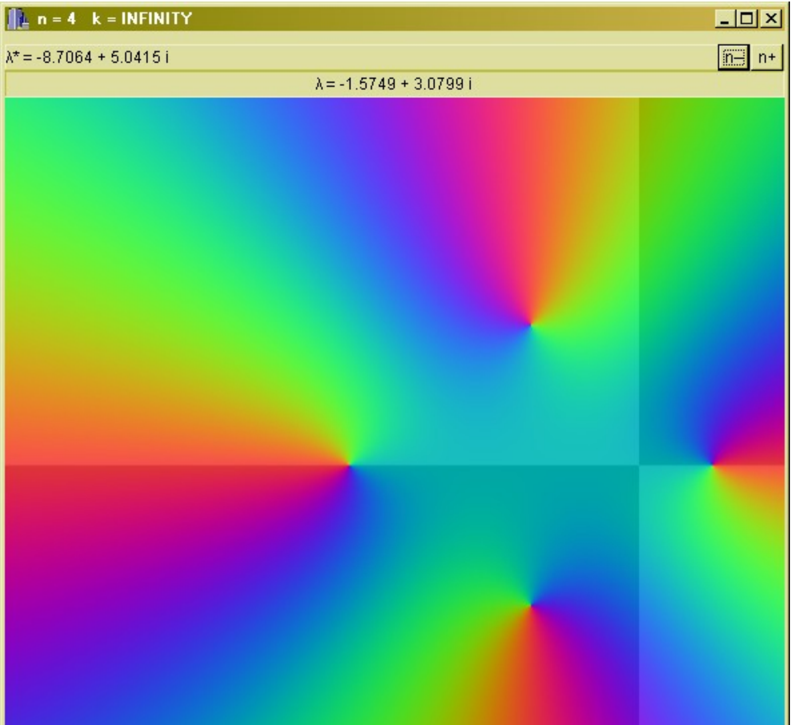
$$\frac{dV}{dt} = AV + F(V) + G(t,\, V),$$

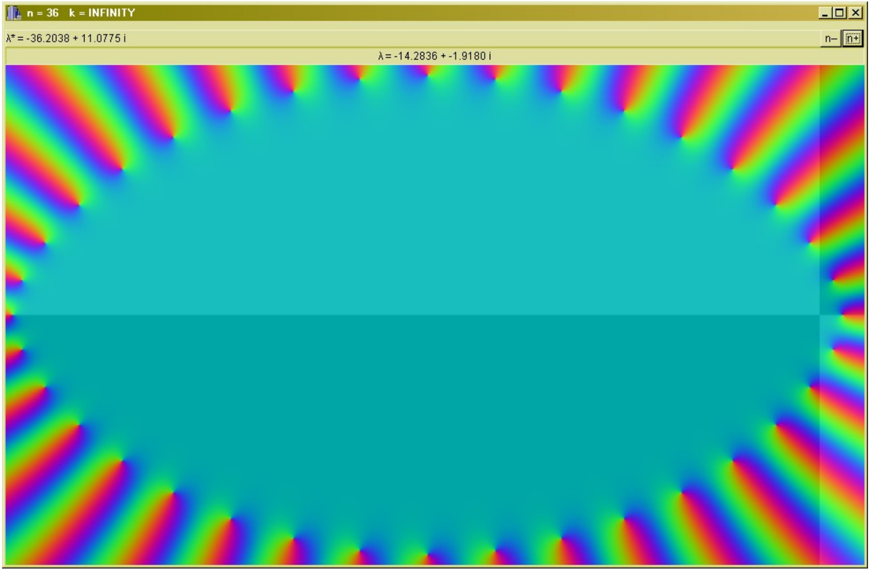
where $V(t)$ is a vector function with components $V_j = \dfrac{d^j v}{dt^j}$, $j = 0, \ldots, n-1,$ A is a constant $n \times n$ matrix with eigenvalues satisfying equation (8), F and G are vector functions with all zero components but the last ones equal to

$$\begin{aligned} F_{n-1}(V) &= p_o \cdot \Big((C+V_0)^k - C^k - kC^{k-1} V_0 \Big), \\ G_{n-1}(t, V) &= (C+V_0)^k \Big(\tilde{p}(t,\, V_0, \ldots, V_{n-1}) - p_0 \Big). \end{aligned}$$


n = 12 k = 152.0000
λ* = -12.2150 + 4.5450 i
n–
λ = -2.5263 + -1.2532 i


n = 3 k = INFINITY
λ* = -8.7064 + 5.0415 i
n–
n+
λ = -6.2315 + 4.9132 i


n = 4   k = INFINITY
λ* = -8.7064 + 5.0415 i
n–
n+
λ = -1.5749 + 3.0799 i


n = 36   k = INFINITY
λ* = -36.2038 + 11.0775 i
n–
n+
λ = -14.2836 + -1.9180 i

Astashova I.V. : Qualitative properties of solutions to quasilinear ordinary differential equations. In: Astashova I.V (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, pp. 22-290. M.: UNITY-DANA (2012) 647 p. (Russian)

I. Astashova. On power and non-power asymptotic behavior of positive solutions to Emden-Fowler type higher-order equations//Advances in Difference Equations. 2013, 2013:220. http://www.advancesindifferenceequations.com/content/2013/1/220

Computer calculations give approximate values of α providing equation (8) to have a pure imaginary root λ. They are, with corresponding values of k, as follows:

if $n = 12$, then $\alpha \approx 0.56,\ k \approx 22.4$;

if $n = 13$, then $\alpha \approx 1.44,\ k \approx 10.0$;

if $n = 14$, then $\alpha \approx 2.37,\ k \approx 6.9$.

Consider the equation

$$y^{(n)} + p_0 \, |y|^k \, \operatorname{sign} y = 0, \tag{11}$$

$n \geq 2,\ k > 1, \quad p_0 \neq 0.$

Hereafter we use the notation

$$\alpha = \frac{n}{k-1}. \tag{12}$$

Theorem

For any integer $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function

$$y(x) = p_0^{\frac{1}{k-1}} (x^* - x)^{-\alpha} h\left(\log(x^* - x)\right), \quad -\infty < x < x^*, \tag{13}$$

is a solution to equation (11).

Corollary

For any integer even $n > 2$ *and real* $k > 1$ *there exists a non-constant oscillatory periodic function* $h(s)$ *such that for any* $p_0 > 0$ *and* $x^* \in \mathbb{R}$ *the function*

$$y(x) = p_0^{\frac{1}{k-1}} (x - x^*)^{-\alpha} h\left(\log(x - x^*) \right), \quad x^* < x < \infty, \qquad (14)$$

is a solution to equation (11)*.*

Corollary

For any integer odd $n > 2$ *and real* $k > 1$ *there exists a non-constant oscillatory periodic function* $h(s)$ *such that for any* $p_0 < 0$ *and* $x^* \in \mathbb{R}$ *the function*

$$y(x) = |p_0|^{\frac{1}{k-1}} (x - x^*)^{-\alpha} h\left(\log(x - x^*) \right), \quad x^* < x < \infty, \qquad (15)$$

is a solution to equation (11)*.*

(with V.Rogachev)

Theorem

For any integer $m \geq 2$ *and even* $n > 2$, *and any real* $k > 1$, $p_0 > 0$, $-\infty < a < b < +\infty$, *equation* (11) *has a solution defined on the segment* $[a, b]$, *vanishing at its end points* a *and* b, *and having exactly* m *zeros on the segment* $[a, b]$.

Theorem

For any integer $m \geq 2$ *and odd* $n > 2$, *и and any real* $k > 1$, $p_0 \neq 0$, $-\infty < a < b < +\infty$, *equation* (11) *has a solution defined on the segment* $[a, b]$, *vanishing at its end points* a *and* b, *and having exactly* m *zeros on the segment* $[a, b]$.

Theorem

For any integer $n > 2$ *and real* $k > 1$, $p_0 > 0$, $-\infty < a < b < +\infty$, *equation* (11) *has a solution defined on the half-open interval* $[a, b)$, *vanishing at its end point* a *and having a countable number of zeros on the interval* $[a, b)$.

Theorem

For any integer odd $n > 2$ *and real* $k > 1$, $p_0 < 0$, $-\infty < a < b < +\infty$, *equation* (11) *has a solution defined on the half-open interval* $(a, b]$, *vanishing at its end point* b *and having a countable number of zeros on the interval* $(a, b]$.