Existence and uniqueness of positive traveling fronts in reaction-diffusion equations with spatio-temporal delays

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Introduction

We are interested in the time-delayed reaction-diffusion equations:

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w) g(u(t-s, x-w)) dw ds$$

- $x \in \mathbb{R}$ is the spatial variable,
- t is the time,
- $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$, g is called birth function.
- $K \geq 0$ and $K \in L^1(\mathbb{R}_+ \times \mathbb{R})$.

Particular cases

- In the case, when $f(u) = u$ and $K(s, w) = \delta(s - h)\delta(w)$, with $h > 0$, we obtain the local reaction-diffusion equations with delay

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x))$$

- when $f(u) = u$ and $K(s, w) = \delta(s - h)K(w)$ with $h > 0$, we obtain the non-local reaction-diffusion equations with delay

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + \int_{\mathbb{R}} K(x - w)g(u(t - h, w))dw$$

- These equations, with appropriate f, g and K, are intensively studied for the last decade.
- They are used to model many ecological and biological processes, where wave phenomena are observed and which depend not only on the present state but also on some past occurrences.
- In a biological context, u is the size of an adult population, so we will consider only positive solutions, and h is the age of maturity.

The main goals:

- the existence of wavefronts solutions,
- the uniqueness (up to translation) of wavefront and semi-wavefront solutions, and
- the minimal speed of propagation of positive travelling wave solutions.

Travelling wave

Definition

- *A travelling wave solution is a bounded positive continuous non-constant* $u(x,t) = \varphi(x+ct)$*, propagating with speed* c*.*
- *In the event that the profile* φ *satisfies the boundary conditions* $\varphi(-\infty) = 0$ *and* $\varphi(+\infty) = \kappa$, $\kappa > 0$*, the travelling wave solution is called a wavefront.*
- *If* φ *satisfies* $\varphi(-\infty) = 0$ *or* $\varphi(+\infty) = 0$*, is called semi-wavefront.*

Example

Consider the diffusive logistic equation (1937)

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)(1-u(t,x)),\ u \geq 0,\ x \in \mathbb{R}.$$

If $u(x,t) = \varphi(x+ct)$ *is a wavefront solution, then the profile* φ *is positive and satisfies*

$$\varphi''(s) - c\varphi'(s) + \varphi(s)(1-\varphi(s)) = 0,$$

$$\varphi(-\infty) = 0,\ \varphi(+\infty) = 1.$$

Kolmogorov A., Petrovskii I., Piskunov N. (Byul. Mosk. Gos. Univ. Ser. A Mat. Mekh.)

The diffusive logistic equation

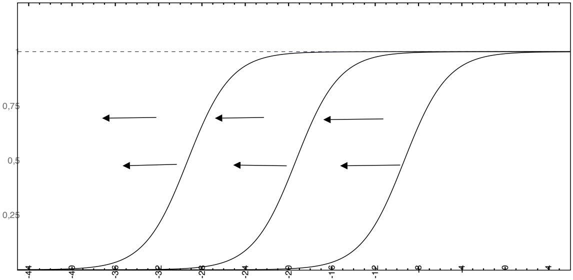


Figure : the profile φ with $c > 0$

- In 1940 Nicholson made pioneer study on the distribution of blowflies population. Nicholson introduced the equation

$$u_t(t,x) = -\delta u(t,x) + pu(t-h,x)e^{-bu(t-h,x)}, t \in \mathbb{R},\ x \in \mathbb{R},\ \delta > 0$$

- In 1977 Mackey & Glass (Science 197) proposed the model hematopoiesis (blood cell production)

$$u_t(t,x) = u(t,x) + \frac{pu(t-h,x)}{1+(u(t-h,x))^n},\ h,p > 0,\ n > 1.$$

- In 1998 So & Yang (J. Differ. Equ.)

$$u_t(t,x) = u_{xx}(t,x) - \delta u(t,x) + pu(t-h,x)e^{-bu(t-h,x)}$$

Typical wavefronts of the monostable reaction-diffusion equation with delay

- Monotone birth function

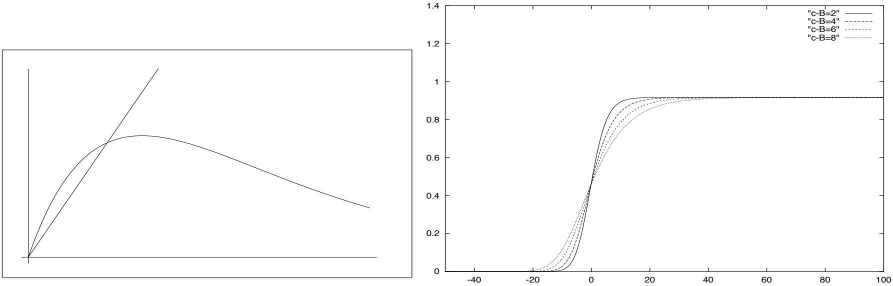
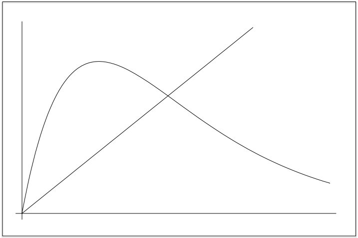


Figure : $g(w) = pwe^{-aw}, p = 2; a = 1$ (Gourley et al., J. Math. Sci., 2004)

Typical wavefronts of the monostable reaction-diffusion equation with delay

- non-monotone birth function



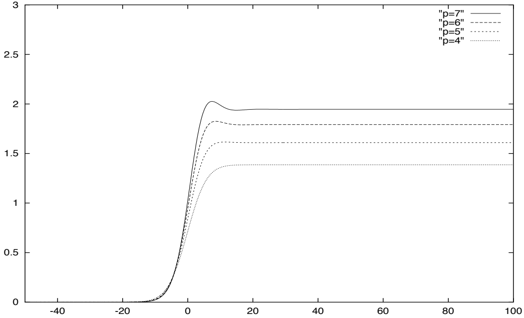


Figure : $g(w) = pwe^{-aw}, p = 4, 5, 6, 7; a = 1$ (Gourley et al., J. Math. Sci., 2004).

Typical wavefronts of the monostable reaction-diffusion equation with delay

- non-monotone birth function

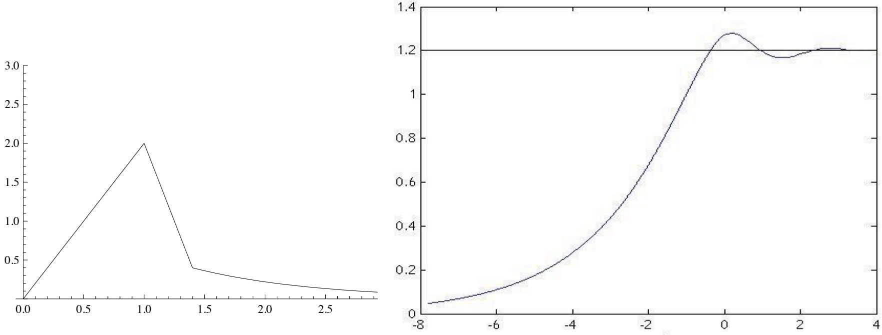


Figure : (Trofimchuk *et al.*, J. Differ. Eq., 2008).

The critical traveling wave

- Corresponds to the slowest travelling wave.

If $g(s) \le g'(0)s$, then the minimal wave speed c_* is defined as the minimal value of c for which the characteristic function associated with the linearization along the trivial equilibrium

$$\chi(z,c) := z^2 - cz - f'(0) + g'(0) \int_0^\infty \int_{\mathbb{R}} K(s,w) e^{-z(cs+w)} dw ds, \; z \in \mathbb{C},$$

has at least one positive zero.

- If g does not dominated by $g'(0)s$, then c_* is obtained by variational principle (Hadeler(1998), Benguria and Depassier(2002)); Accelerating wave (Garnier, 2010).

General Theory

Aiming to prove the uniqueness of semi-wavefronts to a broad family of monostable equations, we develop a version of the fundamental Diekmann-Kaper (D-K) theory of a nonlinear convolution equation for the scalar integral equation

$$\varphi(t) = \int_X d\mu(\tau) \int_{\mathbb{R}} \mathcal{K}(s,\tau) g(\varphi(t-s),\tau) ds, \quad t \in \mathbb{R}.$$

- (X,μ) will denote a measure space with finite measure μ.
- The kernel $\mathcal{K} : \mathbb{R} \times X \to [0,+\infty)$ is integrable with $\int_{\mathbb{R}} \mathcal{K}(s,\tau) ds > 0,\ \tau \in X$.
- The measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, is continuous in φ for every fixed $\tau \in X$.

Examples: nonlinear convolution D-K equation

- When X is just a single point (i.e. $\#X = 1$), we obtain the nonlinear convolution equation

$$\varphi(t) = (g \circ \varphi) * K(t)$$

Diekmann O. and Kaper H.(1978) in several papers proved the existence and uniqueness of wavefronts for g satisfies the subtangential Lipschitz condition $|g(u) - g(v)| \leq g'(0)|u - v|$ for all $c > c_*$.

Problems

- the Diekmann-Kaper uniqueness theorems do not apply to the critical fronts (when $\chi(z, c_*) = \chi'(z, c_*) = 0$).
- the subtangetial Lipschitz condition $|g(s) - g(t)| \leq g'(0)|t - s|$ is not necessary for the uniqueness.
- Some local and non-local reaction-diffusion equations with delay can be write of this form.

Example

Consider the travelling wave solution $u(t,x) = \varphi(x+ct)$, *to equation*

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)),\ x \in \mathbb{R}.$$

- Profile φ solves the delay differential equation

$$\varphi''(t) - c\varphi'(t) - \varphi(t) + g(\varphi(t-hc)) = 0,$$

- Being φ a positive bounded solution, it should satisfy the integral equation

$$\varphi(t) = \frac{1}{\sigma(c)}\Big(\int_{-\infty}^{t} e^{\nu(t-s)} g(\varphi(s-ch))ds + \int_{t}^{+\infty} e^{\mu(t-s)} g(\varphi(s-ch))ds$$

Finally, we get the D-K equation

$$\varphi(t) = \mathcal{K} * g(\varphi)(t),$$

where

$$\mathcal{K}(s) = \frac{1}{\sigma(c)} \begin{cases} e^{\nu(s-ch)}, & s \geq ch \\ e^{\mu(s-ch)}, & s < ch \end{cases} .$$

Example

The nonlocal reaction-diffusion equation with distributive time delays.

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w) g(u(t-s, x-w)) dw ds$$

If $X = \{\tau_1, \tau_2\}$ and

$$\mathcal{K}(s,\tau) = \begin{cases} (N * k_h)(s), & \tau = \tau_1, \\ N, & \tau = \tau_2, \end{cases} \qquad g(s,\tau) = \begin{cases} g(s), & \tau = \tau_1, \\ f_\beta(s), & \tau = \tau_2. \end{cases},$$

where $k_h(w) = K(w - ch)$, $f_\beta(s) = \beta s - f(s)$ and

$$N(s) = \sigma^{-1}(c) \begin{cases} e^{\nu s}, & s \geq 0, \\ e^{\mu s}, & s < 0, \end{cases}$$

- Then we obtain the following equation to perfil φ:

$$\varphi(t)=\frac{1}{\sigma(c)}\left(\int_{-\infty}^{t} e^{\nu(c)(t-s)}(\mathcal{G}\varphi)(s)ds+\int_{t}^{+\infty} e^{\mu(c)(t-s)}(\mathcal{G}\varphi)(s)ds\right),$$

where $(\mathcal{G}\varphi)(t):=\int_0^\infty \int_{\mathbb{R}} K(s,w)g(\varphi(t-cs-w))dwds+f_\beta(\varphi(t))$.

- Thus φ satisfies

$$\varphi''(t)-c\varphi'(t)-f(\varphi(t))+\int_0^\infty \int_{\mathbb{R}} K(s)g\left(\varphi(t-cs-w)dwds\right)ds=0.$$

- Finally, $u(x,t)=\varphi(x+ct)$ is a wave solution to

$$u_t(t,x)=u_{xx}(t,x)-f(u(t,x))+\int_0^\infty \int_{\mathbb{R}} K(s,w)g(u(t-s,x-w))dwds$$

Some works:

- Thieme H., Zhao X.-Q. (J. Differential Equations, 2003)
- Fang J., Zhao X. (J. Differential Equations, 2010)
- Wu S., Liu S. (Applied Mathematics Letters, 2009)
- Z. Xu, P. Weng (Acta Mathematca Sinica, English Series, 2013)
- M. Aguerrea, C. Gomez, S. Trofimchuk (Mathematische Annalen, 2012)
- C. Gomez, H. Prado, S. Trofimchuk (submited, 2012)

Other Models

- the nonlocal KPP-Fisher equation:

$$\varphi_t = J * \varphi - \varphi + g(\varphi).$$

  - Schumacher (I. J. Reine Angew. Math., 1980)
  - Carr and Chmaj (Proc. Amer. Math. Soc., 2004)
  - Coville, Dávila and Martínez (J. Differential Equations, 2008).

- The nonlocal lattice equation

$$c\varphi'(x) = D[\varphi(x+1) + \varphi(x-1) - 2\varphi(x)] - d\varphi(x) + \sum_{k\in\mathbb{Z}} \beta(k) g(\varphi(x-k-cr))$$

  - Fang J., Wei J., Zhao X.-Q. (Proc. Amer. Math. Soc., 2010)
  - Guo J.-S., Wu C.-H. (Osaka J. Math. ,2008)
  - Chen X., Fu S.-C., Guo J.-S. (SIAM J. Math. Anal. 2006)
  - Ma, S., Zou, X. (J. Differential Equations, 2005)
  - Zinner B., Harris G., Hudson W. (J. Differential Equations,1993)

Some Uniqueness Results

- Trofimchuk *et al* (J. Differential Equations, 2008).
- M. Aguerrea, S. Trofimchuk and G. Valenzuela (Proc. R. Soc. A,2008)
- M. Aguerrea, C. Gomez, S. Trofimchuk, (Mathematische Annalen, 2012).
- M. Aguerrea (submited, 2013)

Theorem

If g satisfies the condition
$|g(s_1) - g(s_2)| \leq L|s_1 - s_2|, \;\; s_1, s_2 \geq 0,$ *for some* $L > 0,$ *then equation*

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w)g(u(t-s,x-w))dwds$$

has at most one (modulo translation) semi-wavefront solution $u(x,t) = \varphi(x+ct)$ *for each* $c \geq c_\star$.

Definition

Let $c_\star$ be the minimal value of c for which

$$\chi_L(z,c) := z^2 - cz - \inf_{s\geq 0} f'(s) + L\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-z(cs+w)}dwds,\ L \geq g'(0)$$

has at least one positive root.

We observe that $c_\star \geq c_*$.

Assumptions

- *$g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is such that $g(0) = 0$, $g(s) > 0$ for all $s > 0$, and differentiable at 0 with $g'(0) > 0$.*
- *$f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f(0) = 0$, is strictly increasing with $f'(0) < g'(0)$.*
- *$g, f \in C^{1,\alpha}$ in some neighborhood of 0, with $\alpha \in (0, 1)$.*

The result is obtained by using our generalization of the Diekman-Kaper theory.

- Following to Mallet-Paret (J. Dynam. Differential Equations, 1999), we obtained asymptotic representations of the profile φ.
- $\varphi(t + m) = (a - t)^k e^{\lambda_l t} + e^{(\lambda_l + \delta)t} r(t)$, *with continuous* $r \in L^2(\mathbb{R})$, for some appropriate $a, m \in \mathbb{R}$, $\delta > 0$. Here $k = 0$ [respectively, $k = 1$] if λ_l is a simple [respectively, double] root of $\chi(z) = 0$.

Some Existence Results

- T.Faria, S.Trofimchuk (J. Differential Equations, 2006)
- E.Trofimchuk, P.Alvarado, S.Trofimchuk (J. Differential Equations, 2009)
- M. Aguerrea (Nonlinear Analysis, 2010)
- C. Gomez, H. Prado, S. Trofimchuk (submited, 2012)

Theorem

(i) *If* $g(s) \leq Ls$, $f(s) \geq f'(0)s$ *for all* $s \geq 0$ *and some* $L > 0$, *then the equation*

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w) g(u(t-s, x-w)) dw ds$$

has a semi-wavefront solution $u(x,t) = \varphi(x+ct)$ *propagating with speed* $c \geq c_\star$.

(ii) *for any* $c < c_*$, *there not are a semi-wavefront solution propagating with speed c.*

Theorem

(iii) *if equation $f(s) = g(s)$ has only two solutions: 0 and κ, with κ being globally attracting with respect to $f^{-1} \circ g$, then there is at least one wavefront $u(x,t) = \varphi(x + ct)$ propagating with speed $c \geq c_\star$ such that $\varphi(+\infty) = \kappa$.*

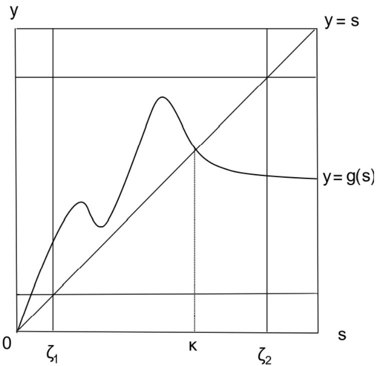


Figure : $G(s) := (f^{-1} \circ g)(s)$

Assumptions

- *$g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is such that $g(0) = 0$, $g(s) > 0$ for all $s > 0$, and differentiable at 0 with $g'(0) > 0$.*
- *$f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is strictly increasing with $f(0) = 0$, $0 < f'(0) < g'(0)$, and further $f(+\infty) > \sup_{s \geq 0} g(s)$.*

We apply the theory developed in C. Gomez, H. Prado, S. Trofimchuk , to prove the existence.

- Dichotomy principle: $\lim_{t \to -\infty} \varphi(t) = 0$ and $\lim_{t \to +\infty} \varphi(t) > \xi > 0$.
- The operator $A\varphi(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) g(\varphi(t - s), \tau) ds$ is completely continuous map on some appropriate space.
- Shauder's fixed point theorem implies the existence of semi-wavefront solution.

We apply the uniqueness results to some non-local reaction-diffusion epidemic and population models with distributed time delay, studied in several works.

- J. Fang, J. Wei, X. Zhao (Spatial dynamics of a nonlocal and time-delayed reaction-diffusion system, Journal of Differ. Equations, 2008)
- S. A. Gourley, Y. Kuang (Wavefronts and global stability in time-delayed population model with stage structure, 2003)
- H. R. Thieme, X.-Q. Zhao (Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, J. Differential Equations, 2003)
- D. Xu, X. Zhao (Asymptotic speed of spread and traveling wave for nonlocal epidemic model, Discrete and Continuous Dynamical Systems-Series B, 2005)

Bounds for the minimal speed

- Aguerrea & Valenzuela (Nonlinear Oscillations, 2010)

We give constructive upper and lower bounds for the minimal speed of propagation of traveling waves for equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(x-w)g(u(t-h,w))dw,\ x \in \mathbb{R}$$

- $\max\left\{2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \frac{2\sqrt{\ln p}}{1+h}\right\} < c_* < \min\left\{\frac{k_1}{1+h}, \frac{k_2}{h}\right\},\ h \in [0,1],$
- $\max\left\{2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \frac{\sqrt{\ln p}}{h}\right\} < c_* < \min\left\{\frac{k_1}{2}, \frac{k_2}{\sqrt{h}}\right\},\ h \in [1,+\infty).$

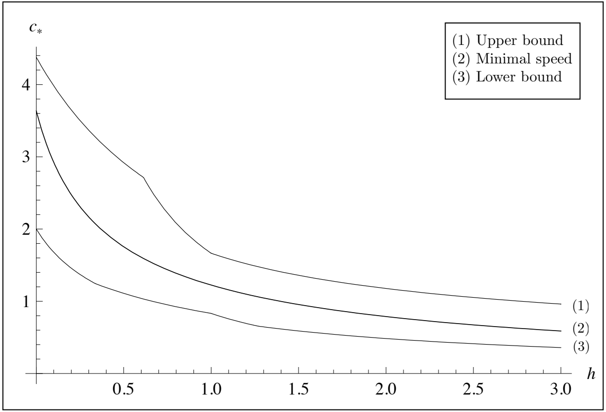


Figure : The minimal speed and its bounds

($g'(0) = 2$, the heat kernel $K_\alpha(s) = (4\pi\alpha)^{-1/2} \exp\left(-s^2/(4\alpha)\right)$, $\alpha = 1$).

Results achieved in these works

- to consider new types of models which include e.g. the nonlocal KPP-Fisher equations (with either symmetric or anisotropic dispersal), nonlocal lattice equations and delayed reaction-diffusion equations;

- to include the critical case (which corresponds to the slowest wavefronts) into the consideration;

- to weaken or to remove various restrictions on kernels and nonlinearities, including the subtangential Lipschitz condition $|g(u) - g(v)| \leq g'(0)|u - v|$ to uniqueness of wave solution.

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Thank very much.