ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH REGULARLY VARYING NONLINEARITIES

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Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\longrightarrow]0, +\infty[$ - is a continuous function, $\varphi_j : \triangle_{Y_j} \longrightarrow]0, +\infty[$ $(j = \overline{0, n-1})$ - continuous regularly varying at $y^{(j)} \longrightarrow Y_j$ functions of orders σ_j, $-\infty < a < \omega \le +\infty$, $\triangle_{Y_j}$ is one-sided neighbourhood of Y_j, Y_j is either 0, or $\pm\infty$.

Measurable function $\varphi : \Delta_Y \longrightarrow]0, +\infty[$, where Y is either 0, or $\pm\infty$ and Δ_Y is one-sided neighbourhood of Y, is regularly variyng at $y \to Y$ (see [1]), if there exists number $\sigma \in \mathbb{R}$ such that

$$\lim_{\substack{y \to Y \\ y \in \Delta Y}} \frac{\varphi(\lambda y)}{\varphi(y)} = \lambda^\sigma \quad \text{for any} \quad \lambda > 0.$$

In this case the number σ is called the order of regularly varying function. Regularly varying as $y \to Y$ zero-order function is called slowly varying function. Every regularly varying as $y \longrightarrow Y$ function of order σ has the representation

$$\varphi(y) = |y|^\sigma L(y), \tag{2}$$

where $L : \Delta_Y \longrightarrow]0, +\infty[$ – is a slowly varying function as $y \to Y$.

Moreover, let's assume that a slowly varying function $L : \Delta_Y \longrightarrow]0, +\infty[$ satisfies *Condition* S_0, if

$$L\left(\nu e^{[1+o(1)] \ln |y|}\right) = L(y)[1 + o(1)] \quad \text{if} \quad y \to Y \quad (y \in \Delta_Y),$$

where $\nu = \operatorname{sign} y$.

For example, the following functions are slowly varying as $y \to Y$:

$$|\ln |y||^{\gamma_1}, \quad \ln^{\gamma_2} |\ln |y||, \quad \gamma_1, \gamma_2 \in \mathbb{R}, \quad \exp(|\ln |y||^{\gamma_3}), \quad 0 < \gamma_3 < 1, \quad \exp\left(\frac{\ln |y|}{\ln |\ln |y||}\right);$$

they have a nonzero finite limit as $y \to Y$.

Among these functions, the first two satisfy the *Condition* S_0, as well as many other functions.

By the definition of a regularly varying functions the differential equation (1) is asymptotically close at $y^{(j)} \longrightarrow Y_j$ $(j = \overline{0, n-1})$ to the equation

$$y^{(n)} = \alpha_0 p(t) \prod_{j=0}^{n-1} |y^{(j)}|^{\sigma_j}.$$

Asymptotic behavior of solutions of this equation is investigated in [2]-[6], as well as in many other works.

In case of regularly varying nonlinearities which are distinct from powers, asymptotic representation of solutions were established at $n = 2$ in works [7]-[11] and at $n \ge 2$, $\varphi_j(y^{(j)}) \equiv 1$ $(j = \overline{1, n-1})$ in [12].

Definition 1. *Solution y of the equation (1) will be called $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$ - solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on an interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions*

$$y^{(j)}(t) \in \Delta_{Y_j} \quad at \quad t \in [t_0, \omega[, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \quad (j = \overline{0, n-1}), \quad \lim_{t \uparrow \omega} \frac{\left[y^{(n-1)}(t)\right]^2}{y^{(n)}(t) y^{(n-2)}(t)} = \lambda_0.$$

Our purpose is to determine the conditions for existence of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$ - solution of the equation (1) at all possible values of λ_0 and asymptotic representations as $t\uparrow\omega$ for such solutions and their derivatives up to and including $n-1$ order.

Assume that numbers ν_j $(j=\overline{0,n-1})$, determined by

$$\nu_j=\begin{cases} 1, & \text{if either } Y_j=+\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is right neighborhood of } 0,\\ -1, & \text{if either } Y_j=-\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is left neighborhood of } 0,\end{cases}$$

are like

$$\nu_j\nu_{j+1}>0 \text{ with } Y_j=\pm\infty \text{ and } \nu_j\nu_{j+1}<0 \text{ with } Y_j=0 \quad (j=\overline{0,n-2}). \tag{2}$$

Such conditions for ν_j $(j=\overline{0,n-1})$ are necessary for solutions of (1), which are determined in left neighborhood of ω and which satisfy first two conditions of the definition 1.

Let's reduce two of the theorems established for the equation (1), concerning $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)-$ solutions in a case, when $\lambda_0\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1},1\right\}$.

Let's define

$$a_{0i}=(n-i)\lambda_0-(n-i-1) \quad (i=1,\dots,n),$$

$$\gamma_0=1-\sum_{j=0}^{n-1}\sigma_j,\quad \mu_n=\sum_{j=0}^{n-2}\sigma_j(n-j-1),\quad C=\prod_{j=0}^{n-2}\left|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\right|^{\sigma_j},$$

$$\pi_\omega(t)=\begin{cases} t, & if \quad \omega=+\infty,\\ t-\omega, & if \quad \omega<+\infty,\end{cases}\qquad J_n(t)=\int_{A_n}^{t}p(\tau)|\pi_\omega(\tau)|^{\mu_n}\,d\tau,$$

where limit of integration A_n is either a, or ω and is chosen so that the integral tends either to $\pm\infty$ or to zero as $t\uparrow\omega$.

Theorem 1. *Let $\lambda_0\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\dots,\frac{n-2}{n-1},1\right\}$ and $\gamma_0\neq0$. Then for existence of $P_\omega(Y_0,\dots,Y_{n-1},\lambda_0)$ - solutions of equation (1) it is necessary and if algebraic equation*

$$\sum_{j=0}^{n-1}\sigma_j\prod_{i=j+1}^{n-1}a_{0\,i}\prod_{i=1}^{j}(a_{0\,i}+\rho)=(1+\rho)\prod_{i=1}^{n-1}(a_{0\,i}+\rho) \tag{3}$$

does not have roots with zero real part, is sufficiently that inequality (2), inequalities

$$\nu_{0j}\nu_{0j+1}a_{0j+1}(\lambda_0-1)\pi_\omega(t)>0 \quad (j=\overline{0,n-2}),\quad \alpha_0\nu_{n-1}\gamma_0J_n(t)>0 \quad at \quad t\in]a,\omega[$$

and condition

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_n'(t)}{J_n(t)}=\frac{\gamma_0}{\lambda_0-1}$$

are satisfied. Moreover, for each such solution at $t\uparrow\omega$, following asymptotic representations are valid

$$y^{(j)}(t)=\frac{[(\lambda_0-1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}y^{(n-1)}(t)[1+o(1)] \quad (j=0,1,\dots,n-2),$$

$$\frac{\left|y^{(n-1)}(t)\right|^{\gamma_s}}{\prod\limits_{j=0}^{n-1}L_j\left(\frac{[(\lambda_0-1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}y^{(n-1)}(t)\right)}=\alpha_0\nu_{n-1}\gamma_0CJ_n(t)[1+o(1)],$$

where $L_j\left(y^{(j)}\right) = |y^{(j)}|^{-\sigma_j}\varphi_{sj}\left(y^{(j)}\right) \quad (j = \overline{0, n-1})$. *Furthermore, there exists an m-parameter family of such solutions if , among the roots of equation (3), there are m roots (with regard of multiplicities) with the real part having the same sign as the function* $(1-\lambda_0)\pi_\omega(t)$.

Theorem 2. *Let the conditions of theorem 1 be satisfied and the functions* L_j $(j = \overline{0, n-1})$ *satisfy Condition* S_0 . *Then each* $P_\omega(Y_0, \ldots, Y_{n-1}, \lambda_0)$ *- solution of differential equation (1) admits the following asymptotic representations as* $t \uparrow \omega$

$$y^{(j)}(t) \sim \frac{\nu_{n-1}[(\lambda_0 - 1)\pi_\omega(t)]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \left| \gamma_0 C J_n(t) \prod_{i=0}^{n-1} L_i\left(\nu_i |\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_0 - 1}}\right)\right|^{\frac{1}{\gamma_0}} \quad (j = \overline{0, n-1}).$$

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