Time discretization of the SST-generalized Navier-Stokes Equations: Positive and Negative results

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Abstract

In the theory of the Navier-Stokes equations, the proofs of some basic known results, like for example the uniqueness of solutions to the steady Navier-Stokes equations under smallness assumptions on the force or the stability of certain difference schemes actually only use a small range of properties and are therefore valid in a more general context. In the lecture this context is made concrete by means of the concept of an SST space. A real vector space V is called an SST space if it is equipped with two scalar products $(\cdot,\cdot)$ and $((\cdot,\cdot))$ and a trilinear form $(\cdot,\cdot,\cdot)$ such that the norm $||\cdot|| = ((\cdot,\cdot))^{1/2}$ is stronger than the norm $|\cdot| = (\cdot,\cdot)^{1/2}$ and such that the trilinear form is continuous with respect to the norm $||\cdot||$ and skew-symmetric in the last two components. So the acronym SST stands for scalar product, scalar product, trilinear form. In the special case of the Navier-Stokes equations, V is the set of solenoidal vector fields on an open bounded subset of $\mathbb{R}^3$ with zero boundary values, $(\cdot,\cdot)$ is the L^2 scalar product and $((\cdot,\cdot))$ the Dirichlet scalar product $(\nabla\cdot,\nabla\cdot)$. Other examples of SST spaces are introduced and serve as counterexamples to disprove uniqueness and stability conjectures which are open questions in the special case of the Navier-Stokes equations. Each time such a counterexample is invoked it proves that the corresponding statement does not hold in SST spaces in general. Of course, in each such situation the corresponding statement might nevertheless be true for the special case of the Navier-Stokes equations and the counterexample does not answer the above mentioned open question. But it shows that it is impossible to prove the corresponding statement for the Navier-Stokes equations when using only the tools available in SST spaces. These tools are not as weak as one might believe: The above-mentioned basic known uniqueness and stability results, as well as several other uniqueness results new to our knowledge are proven for general SST spaces.